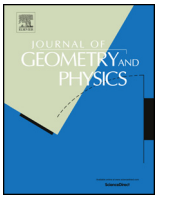
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The nonholonomic bracket on contact mechanical systems



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ABSTRACT

In this paper, we study contact nonholonomic mechanical systems. The contribution of the paper could be divided into two blocks. Firstly, we present a general framework for contact constrained dynamics in such a way that the constraints are geometrically described by a submanifold of the contact manifold and the reaction forces are given by a distribution along the constraint submanifold. In this framework, we describe the constrained differential equations, examine the conditions for the existence and uniqueness of solutions of these equations, and construct two different brackets of functions which describe the evolution of the system. We also prove that nonholonomic contact Lagrangian systems are particular cases of the above general framework. In addition, this general framework permits us to develop the Hamiltonian counterpart and, in this setting, we present the second main contribution of the paper: the construction of another bracket, which is a natural extension of that defined by R.J. Eden, but now it is an almost Jacobi bracket since it does not satisfy the Leibniz rule.

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1. Introduction

The study of nonholonomic mechanics has a long-standing history (see historical reviews in [15] and [7,6]), but they undergo a spectacular development after its description in geometrical terms (see [42,33,5,4,20,3]. Indeed, the use of the tools of differential geometry is a turning point, just as happened in particle mechanics with the use of symplectic geometry and Lie groups (see [2,1,35,21]). The book [38] is also an extraordinary source of examples and practical applications of nonholonomic systems, and is always a mandatory reference on the subject.

One of the most relevant constructions in the case of nonholonomic systems is a bracket which allows us to obtain the evolution of an observable; this bracket will obviously be non-integrable, i.e., an almost-Poisson bracket. This type of bracket was first considered in two pioneering papers by Eden [26,27], and described more clearly in a paper by van der Schaft and Maschke [41]. Later, de León et al. [11] defined in an intrinsic way a nonholonomic bracket which turned out to coincide on the constraint submanifold; to do that, they used two alternative symplectic decompositions of the tangent bundle of the phase space, constructed independently by de Leon et al. [11] and Bates and Sniatycki [3]. Recently, de León et al. [25] proved that it also coincides with what could be called Eden's bracket.

These considerations refer to nonholonomic systems in symplectic environments, both in the case of Lagrangian and Hamiltonian descriptions. Of course, the time-dependent case has also been treated using cosymplectic geometry. But recently a great interest has arisen in the case of contact Hamiltonian systems, which reflect a dissipative nature as opposed to the conservative nature of symplectic systems [22,9,8,16,18,19]. This has led to study (we believe for the first time) the case of contact nonholonomic [24] systems, in which apart from obtaining the equations of motion, a nonholonomic bracket is defined which is almost Jacobi, i.e., it satisfies neither satisfies the Jacobi identity nor the Leibniz identity.

The contributions of the present paper may essentially be summarized into two main blocks. The first contribution is *to construct a general framework for contact constrained dynamics* by considering a contact manifold (P, η), a constrained submanifold $M \subseteq P$ and a distribution F on P encoding the reaction forces. Here, we describe the associated constrained differential equations, and extensively study the uniqueness and existence of solutions of these equations. In fact, it should be noticed that we present a specific way of projecting the solutions of the unconstrained equation into solutions of the constrained equation. In this general framework, we were able, in addition, to construct two different brackets, and we study the relation between them. The first one, the nonholonomic bracket, is given by an almost Jacobi structure, and it provides the evolution of observables. For the second one, the P-nonholonomic bracket, we define a new projection P in such a way that, under some conditions, it also projects the unconstrained dynamics onto the constrained dynamics. This projection gives rise to a different almost Jacobi structure which induces the P-nonholonomic bracket. Finally, we prove that both brackets coincide on the constraint submanifold. Furthermore, we compare this general framework with the previously one defined for contact Lagrangian dynamics [24], and we prove that the constrained dynamics induced by a Lagrangian on a manifold Q, subjected to a regular distribution of kinematic constraints, is just a particular case.

We take advantage of this development made in such a general context to develop the Hamiltonian counterpart of nonholonomic systems from the Lagrangian side, via the Legendre transformation. Thus, we describe in detail the nonholonomic bracket for the constrained Hamiltonian system, and its associated almost Jacobi structure.

The second contribution of the paper is, essentially, *to extend the Eden construction in the case of mechanical systems* [26,27], that is, when the kinetic energy comes from a Riemannian metric, and so, we obtain a bracket *à la* Eden that coincides with the previous ones. This opens new perspectives to study the possible quantization of these systems, which has been outlined in the symplectic case. Indeed, this was the motivation for Eden's pioneering work under Dirac's supervision.

The paper is structured as follows. Section 2 is devoted to give an introduction to contact Hamiltonian systems, in order that the paper will be as self-contained as possible. In Section 3 we develop a general framework for nonholonomic systems, in the sense that we consider a contact manifold, a constraint submanifold and a bundle of forces acting on the system. Then, we obtain conditions for the existence and uniqueness of dynamical solutions of the corresponding equations. Section 4 is devoted to obtain two nonholonomic brackets using two different decompositions of the tangent bundle of the phase space, and we prove that both coincide. In Section 5 we use the above procedure just to the Lagrangian case and, thus, we re-obtain the almost Jacobi bracket given in [24]. In Section 6 we apply the general scheme developed in Section 3 to the case of contact Hamiltonian nonholonomic systems, and, in the particular case when the kinetic energy comes from a Riemannian metric, we define the corresponding Eden bracket, which coincides with the previous ones. Finally, in Section 7 we illustrate the results of the paper by calculating the nonholonomic bracket in an example: a sleigh with homogeneous and isotropic Rayleigh dissipation.

2. Contact Hamiltonian systems

In this section, we will give a brief introduction to contact structures and contact Hamiltonian systems. We will start with a more general geometric structure, the so-called *Jacobi manifolds* [32,36,35].

Definition 1. A *Jacobi manifold* is a triple (M, Λ, E), where Λ is a bivector field (namely, a skew-symmetric contravariant 2-tensor field) and $E \in \mathfrak{X}(M)$ is a vector field, such that the following identities are satisfied:

$$[\Lambda, \Lambda] = 2E \wedge \Lambda \tag{1}$$

$$\mathcal{L}_E \Lambda = [E, \Lambda] = 0, \tag{2}$$

where $[\cdot, \cdot]$ is the Schouten–Nijenhuis bracket.

By using the Jacobi bivector Λ, we may induce a vector bundle morphism between T^*M and TM.

$$\begin{aligned} \sharp_\Lambda : T^*M &\to TM \\ \alpha &\mapsto \Lambda(\alpha, \cdot). \end{aligned} \tag{3}$$

On the other hand, the Jacobi structure induces (and it is characterized by) a Lie bracket on the space of functions $\mathcal{C}^\infty(M)$, which is called *Jacobi bracket*.

Definition 2. A *Jacobi bracket* $\{\cdot, \cdot\}$ on a manifold M is a map $\{\cdot, \cdot\} : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ that satisfies

1. $(\mathcal{C}^\infty(M), \{\cdot, \cdot\})$ is a real Lie algebra. That is, $\{\cdot, \cdot\}$ is $\mathbb{R}$-bilinear, antisymmetric and satisfies the Jacobi identity:

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \tag{4}$$

   for arbitrary functions $f, g, h \in \mathcal{C}^\infty(M)$.
2. It satisfies the following locality condition: for any $f, g \in \mathcal{C}^\infty(M)$,

$$\operatorname{supp}(\{f, g\}) \subseteq \operatorname{supp}(f) \cap \operatorname{supp}(g), \tag{5}$$

   where $\operatorname{supp}(f)$ is the topological support of f, where the closure of the set in which f is non-zero.

Namely, $(\mathcal{C}^\infty(M), \{\cdot, \cdot\})$ is a local Lie algebra in the sense of Kirillov [32].

Thus, given a Jacobi manifold (M, Λ, E) we can define a Jacobi bracket as follows

$$\{f, g\} = \Lambda(\mathrm{d}f, \mathrm{d}g) + fE(g) - gE(f). \tag{6}$$

Conversely, for a Jacobi bracket $\{\cdot, \cdot\}$, we may consider the vector field E given by,

$$E(f) = \{1, f\}, \ \forall f \in \mathcal{C}^\infty(M).$$

Then, its associated Jacobi structure is defined by Eq. (6). In fact, we may prove the following result.

Theorem 1 *([17]). Given a manifold M and a $\mathbb{R}$-bilinear map $\{\cdot, \cdot\} : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$, the following assertions are equivalent.*

1. *The map $\{\cdot, \cdot\}$ is a Jacobi bracket.*
2. *$(\mathcal{C}^\infty(M), \{\cdot, \cdot\})$ is a Lie algebra which satisfies the generalized Leibniz rule,*

$$\{f, gh\} = g\{f, h\} + h\{f, g\} + ghE(f), \tag{7}$$

   where E is a vector field on M.
3. *There is a bivector field Λ and a vector field E such that (M, Λ, E) is a Jacobi manifold, and $\{\cdot, \cdot\}$ is given as in Eq. (6).*

Given a function $f \in C^\infty(M)$, its associated *Hamiltonian vector field* X_f is given by

$$X_f = \sharp_\Lambda(df) + fE, \tag{8}$$

where $\sharp_\Lambda : T^*M \longrightarrow TM$ is defined by Eq. (3).

The characteristic distribution $\mathcal{C}$ of a Jacobi manifold is spanned by all the Hamiltonian vector fields, that is,

$$\mathcal{C} = \operatorname{Im} \sharp_\Lambda + \langle E \rangle.$$

As we have mentioned, a particular case of Jacobi structure is provided by a contact form.

Definition 3. A *contact manifold* is a pair (M, η), with M a manifold of dimension $2n + 1$ and η is a 1-form on M in such a way that $\eta \wedge (\mathrm{d}\eta)^n$ is a volume form; then, η will be called a *contact form*.

Notice that, a contact form may be equivalently defined as a non-vanishing 1-form η such that, for each $q \in M$, the restriction $\mathrm{d}\eta_{|Ker(\eta_q)}$, is non-degenerate, i.e., it is a *symplectic form* on the vector bundle $Ker\,\eta$ (see [34]).

Remark 1. Notice that some authors define a contact structure as a smooth distribution D on M such that it is locally generated by contact forms η via their kernels. This definition is not equivalent to ours, and is less convenient for our purposes, as explained in [16]. For a detailed study on contact forms, we recommend [35,40].

Example 1. Consider the canonical coordinates (x^i, y^j, z) on $\mathbb{R}^{2n+1}$, and define the 1-form η on $\mathbb{R}^{2n+1}$ as:

$$\eta = \mathrm{d}z - y^i \mathrm{d}x^i. \tag{9}$$

A simple computation shows that $(\mathbb{R}^{2n+1}, \eta)$ is a contact manifold.

We also have

$$\mathrm{d}\eta = \mathrm{d}x^i \wedge \mathrm{d}y^i. \tag{10}$$

Thus, the family of vector fields $\{X_i, Y_i\}_i$ defined by

$$X_i = \frac{\partial}{\partial x^i} + y^i \frac{\partial}{\partial z}, \quad Y_i = \frac{\partial}{\partial y^i} \tag{11}$$

generates the kernel of η. Furthermore,

$$\mathrm{d}\eta\left(X_i, X_j\right) = \mathrm{d}\eta(Y_i, Y_j) = 0, \quad \mathrm{d}\eta\left(X_i, Y_j\right) = \delta^i_j.$$

In fact, any contact form locally looks like the contact form defined in Example 1.

Theorem 2 *(Darboux theorem). [30] Suppose η is a contact form on a $2n+1$-dimensional manifold M. For each $x \in M$ there are local coordinates (q^i, p_i, z) centered at x such that*

$$\eta = \mathrm{d}z - p_i \, \mathrm{d}q^i. \tag{12}$$

This system of coordinates will be called Darboux coordinates.

We can check that η is a contact form, if and only if, the mapping

$$\begin{aligned} &\flat : TM \to T^*M, \\ &\qquad v \mapsto \iota_v \mathrm{d}\eta + \eta(v)\eta, \end{aligned} \tag{13}$$

is a vector bundle isomorphism. We will also denote by $\flat : \mathfrak{X}(M) \to \Omega^1(M)$ its associated isomorphism of $\mathcal{C}^\infty(M)$-modules. The inverse of $\flat$ will be denoted by $\sharp$.

By using this isomorphism, one may define the *Reeb vector field* $\mathcal{R}$ as the unique vector field on M satisfying that

$$\flat(\mathcal{R}) = \eta$$

Equivalently, $\mathcal{R}$ is the unique vector field on M such that

$$\iota_{\mathcal{R}} \mathrm{d}\eta = 0, \quad \iota_{\mathcal{R}} \eta = 1. \tag{14}$$

Observe that in Darboux coordinates, the Reeb vector field $\mathcal{R}$ is the partial derivative $\frac{\partial}{\partial z}$. In fact,

- $\flat\left(\frac{\partial}{\partial p_i}\right) = -\mathrm{d}q^i$
- $\flat\left(\frac{\partial}{\partial q^i}\right) = \mathrm{d}p_i - p_i \eta$

Then,

- $\sharp\left(\mathrm{d}q^i\right) = -\frac{\partial}{\partial p_i}$
- $\sharp\left(\mathrm{d}p_i\right) = \frac{\partial}{\partial q_i} + p_i \frac{\partial}{\partial z}$
- $\sharp\left(\mathrm{d}z\right) = \frac{\partial}{\partial z} - p_i \frac{\partial}{\partial p_i}$

The notion of *contact Hamiltonian vector field* X_H for a *Hamiltonian function* $H : M \to \mathbb{R}$, is defined by the formula

$$\flat(X_H) = \mathrm{d}H - (\mathcal{R}(H) + H)\,\eta. \tag{15}$$

Then, it is locally expressed as follows

$$X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial H}{\partial q^i} + p_i\frac{\partial H}{\partial z}\right)\frac{\partial}{\partial p_i} + \left(p_i\frac{\partial H}{\partial p_i} - H\right)\frac{\partial}{\partial z} \tag{16}$$

Therefore, its integral curves satisfy the *contact Hamilton equations*

$$\begin{cases} \dfrac{\mathrm{d}q^i}{\mathrm{d}t} &= \dfrac{\partial H}{\partial p^i} \\ \dfrac{\mathrm{d}p_i}{\mathrm{d}t} &= -\dfrac{\partial H}{\partial q^i} - p_i\dfrac{\partial H}{\partial z} \\ \dfrac{\mathrm{d}z}{\mathrm{d}t} &= p_i\dfrac{\partial H}{\partial p_i} - H \end{cases} \tag{17}$$

We may define the Jacobi structure (M, Λ, E) associated to the contact manifold (M, η) as follows

$$\Lambda(\alpha, \beta) = -\mathrm{d}\eta(\sharp(\alpha), \sharp(\beta)), \quad E = -\mathcal{R}. \tag{18a}$$

One can prove that the Hamiltonian vector fields X_f defined with both structures, Eq. (3) for Jacobi or and Eq. (15) for contact, coincide. Equivalently, the notion of contact Hamiltonian vector field X_H for a Hamiltonian function H, can be alternatively obtained by the formula

$$X_H = \sharp_\Lambda(dH) - H\mathcal{R}. \tag{19}$$

Observe that, in the case of contact manifolds, the map $\sharp_\Lambda$ can be written directly in terms of the contact structure [16, Section 3] as:

$$\sharp_\Lambda(\alpha) = \sharp(\alpha) - \alpha(\mathcal{R})\mathcal{R}. \tag{20}$$

Remark 2. It is important do not confuse $\sharp$ (the inverse of $\flat$), with $\sharp_\Lambda$ (the morphism from T^*M to TM associated to Λ).

Using the contact form η, the tangent bundle of M may be decomposed into a Whitney sum as

$$TM = \ker\eta \oplus \ker\mathrm{d}\eta. \tag{21}$$

Furthermore, it is easy to realize that $\ker\mathrm{d}\eta$ is generated by the Reeb vector field $\mathcal{R}$. Notice that, $\sharp_\Lambda$ is not an isomorphism, since

$$\ker\sharp_\Lambda = \langle\eta\rangle$$

Therefore,

$$\sharp_\Lambda\left(T^*M\right) = \ker\eta$$

Thus, the above decomposition (21) may be written in terms of the Jacobi structure Λ.

Definition 4. Let $\mathcal{D}_x \subseteq T_xM$ be a subspace, where $x \in M$. We say that $\mathcal{D}_x$ is

1. *Horizontal* if $\mathcal{D}_x \subseteq \ker\eta_x$.
2. *Vertical* if $\mathcal{R}_x \in \mathcal{D}_x$.
3. *Oblique* otherwise. By a dimensional counting argument, this is equivalent to $\mathcal{D}_x = (\mathcal{D}_x \cap \ker\eta_x) \oplus \langle\mathcal{R}_x + v_x\rangle$, with $v_x \in \ker\eta_x \setminus \mathcal{D}_x$.

We say that a distribution $\mathcal{D}$ is *horizontal*/*vertical*/*oblique* if $\mathcal{D}_x$ is *horizontal*/*vertical*/*oblique*, for every point x, respectively.

For any fixed contact form η, me may define the following 2-tensor,

$$\omega = \mathrm{d}\eta + \eta\otimes\eta. \tag{22}$$

Then, $\flat$ is the contraction of ω (see Eq. (13)).

Definition 5. Let be $\mathcal{D} \subseteq TM$ a distribution on M. Then, we may define the following notions of complement with respect to ω:

$$\begin{aligned} \mathcal{D}^\perp &= \{v \in TM \mid \omega(w, v) = \flat(w)(v) = 0,\ \forall w \in \mathcal{D}\} = (\flat(\mathcal{D}))^\circ, \\ {}^\perp\mathcal{D} &= \{v \in TM \mid \omega(v, w) = 0, \forall w \in \mathcal{D}\} = \flat^{-1}(\mathcal{D}^\circ). \end{aligned} \tag{23}$$

$\mathcal{D}^\perp$ and ${}^\perp\mathcal{D}$ will be called **right** and **left orthogonal complements respectively**.

Observe that, considering ω_- as the 2-tensor associated to the contact form $-\eta$, then we have, $\omega(w, v) = \omega_-(v, w)$, for all $v, w \in TM$ and, therefore,

$$\mathcal{D}_-{}^{\perp} = {}^{\perp}\mathcal{D},$$

where

$$\mathcal{D}_-{}^{\perp} = \{v \in TM \mid \omega_-(w, v) = 0,\ \forall w \in \mathcal{D}\}$$

We have

$${}^{\perp}(\mathcal{D}^{\perp}) = ({}^{\perp}\mathcal{D})^{\perp} = \mathcal{D}. \tag{24}$$

Furthermore, we may interchange sums and intersections, i.e.,

$$(\mathcal{D} \cap \Gamma)^{\perp} = \mathcal{D}^{\perp} + \Gamma^{\perp}, \tag{25a}$$

$$(\mathcal{D} + \Gamma)^{\perp} = \mathcal{D}^{\perp} \cap \Gamma^{\perp}, \tag{25b}$$

for all distributions $\mathcal{D}$ and Γ on M (for the left complement, we obtain identical identities to those in the right case).

Lemma 3 *([24]). Let $\mathcal{D}$ be a distribution on M.*

- *If $\mathcal{D}$ is horizontal, then*

$$\mathcal{D}^{\perp} = {}^{\perp}\mathcal{D} = \{v \in TM \mid \mathrm{d}\eta(v, w) = 0,\ \forall w \in \mathcal{D}\}. \tag{26}$$

  Hence, $\mathcal{D}^{\perp}$ is vertical.
- *If $\mathcal{D}$ is vertical, then*

$$\mathcal{D}^{\perp} = {}^{\perp}\mathcal{D} = \{v \in \ker\eta \mid \mathrm{d}\eta(v, w) = 0,\ \forall w \in (\mathcal{D} \cap \ker\eta)\}. \tag{27}$$

  Hence, $\mathcal{D}^{\perp}$ is horizontal.

In what follows, we will omit the word *contact* and call X_H simply a *Hamiltonian vector field*. The triple (M, η, H) will be called *contact Hamiltonian system*.

Proposition 1. *Let H be a Hamiltonian function. The following statements are equivalent:*

1. *X_H is the Hamiltonian vector field of H.*
2. *It satisfies that*

$$\eta(X_H) = -H, \tag{28a}$$

$$\iota_{X_H}\mathrm{d}\eta_{|\ker\eta} = \mathrm{d}H_{|\ker\eta} \tag{28b}$$

3. *It satisfies that*

$$\eta(X_H) = -H, \tag{29a}$$

$$\mathcal{L}_{X_H}\eta = a\eta, \tag{29b}$$

   for some function a. Notice that this implies that $a = -\mathcal{R}(H)$.

By Cartan's formula, formulas (29a) and (29b) imply the following identity

$$\iota_{X_H}\mathrm{d}\eta = \mathrm{d}H - \mathcal{R}(H)\eta. \tag{30a}$$

Thus, in contrast with the symplectic case, the energy and the phase space volume are not preserved. In fact, denoting the volume form by $\Omega = \eta \wedge (\mathrm{d}\eta)^n$, the *energy and volume dissipation* are given by the following formulas:

i)

$$\mathcal{L}_{X_H}H = -\mathcal{R}(H)H. \tag{31}$$

ii)

$$\mathcal{L}_{X_H}\eta = -\mathcal{R}(H)\eta. \tag{32}$$

iii)

$$\mathcal{L}_{X_H}\Omega = -(n+1)\mathcal{R}(H)\Omega. \tag{33}$$

3. General framework for contact Hamiltonian constrained dynamics

Assume a $2n+1$-dimensional contact manifold (P,η) equipped with a Hamiltonian function $H : P \to \mathbb{R}$. The constraints will be codified into an r-dimensional embedded submanifold M of P, called the *constraint submanifold*. Furthermore, we will consider a regular distribution F on P along M, namely, a vector subbundle of $T_M P$ (the space of tangent vectors to P at points in M). In general, the distribution F characterizes the *reaction forces*, via the annihilator F°.

Thus, we are interested in finding vector fields $X \in \mathfrak{X}(P)$ that are tangent to the constraint submanifold M and satisfy the following equations

$$\flat(X) = \mathrm{d}H - (H + \mathcal{R}(H))\,\eta + \lambda_a \Psi^a,$$

where Ψ^a are reaction forces and λ_a are some Lagrange multipliers to be determined. In other words, we are looking for solutions of the following equations

$$\begin{cases} \flat(X) - \mathrm{d}H + (H + \mathcal{R}(H))\,\eta \in F^\circ \\ X_{|M} \in \mathfrak{X}(M). \end{cases} \tag{34}$$

From now on, we will consider $\left(q^i, p_i, z\right)$ as a (local) system of Darboux coordinates for η. Let $\{\Psi^a = \Phi^a_i \mathrm{d}q^i + \Psi^a_i \mathrm{d}p_i + \mu^a \mathrm{d}z\}$ be a (local) basis of 1-forms of F°, and a vector field Y on P such that

$$\flat(Y) \in F^\circ. \tag{35}$$

Then, there exist some (local) functions λ_a such that

$$\flat(Y) = \lambda_a \Psi^a = \lambda_a \Phi^a_i \mathrm{d}q^i + \lambda_a \Psi^a_i \mathrm{d}p_i + \lambda_a \mu^a \mathrm{d}z.$$

Observe that $\eta = \mathrm{d}z - p_i \mathrm{d}q^i$ and, hence, $\mathrm{d}\eta = \mathrm{d}q^i \wedge \mathrm{d}p_i$. Thus,

$$\begin{aligned} \lambda_a \Psi^a_i &= \{\flat(Y)\}\left(\frac{\partial}{\partial p_i}\right) = \iota_Y \mathrm{d}\eta\left(\frac{\partial}{\partial p_i}\right) + \eta(Y)\,\eta\left(\frac{\partial}{\partial p_i}\right) \\ &= -\iota_{\frac{\partial}{\partial p_i}} \mathrm{d}\eta(Y) = Y\left(q^i\right) \end{aligned}$$

Analogously, contracting by $\frac{\partial}{\partial z}$, we obtain that $\eta(Y) = \lambda_a \mu^a$ and

$$Y(z) = \lambda_a \left[\mu^a + p_i \Psi^a_i\right]$$

Finally,

$$\begin{aligned} \lambda_a \Phi^a_i &= \{\flat(Y)\}\left(\frac{\partial}{\partial q^i}\right) = \iota_Y \mathrm{d}\eta_Q\left(\frac{\partial}{\partial q^i}\right) + \eta_Q(Y)\,\eta_Q\left(\frac{\partial}{\partial q^i}\right) \\ &= -Y(p_i) - \lambda_a \mu^a p_i \end{aligned}$$

Then,

$$Y(p_i) = -\lambda_a \left[\mu^a p_i + \Phi^a_i\right].$$

Now, let us observe that a vector field X solving Eq. (34) should satisfy that $\flat(X - X_H) \in F^\circ$, where X_H is the solution of the contact Hamiltonian system (17). Therefore,

$$\begin{aligned} X = &\left[\frac{\partial H}{\partial p_i} + \lambda_a \Psi^a_i\right]\frac{\partial}{\partial q^i} - \left[\frac{\partial H}{\partial q^P i} + p_i \frac{\partial H}{\partial z} + \lambda_a\left(\mu^a p_i + \Phi^a_i\right)\right]\frac{\partial}{\partial p_i} \\ &+ \left[\left(p_i \frac{\partial H}{\partial p_i} - H + \lambda_a\left(\mu^a + p_i \Psi^a_i\right)\right)\right]\frac{\partial}{\partial z} \end{aligned}$$

The Lagrange multipliers λ_a will be determined from the tangency condition to M, and, taking into account the generality of the model, *they will generally not be unique*. Notice that M may be locally described in terms of independent (local) functions $\{\varphi^a\}$ on P in the following way

$$M = \{x \in P \mid \varphi^a(x) = 0,\ \forall a\} = \varphi^{-1}(0),$$

where $\varphi = \left(\varphi^1, \ldots, \varphi^{2n-1-r}\right)$. These functions φ^a will be called *constraint functions* defining M.

Theorem 4. *Let be a Hamiltonian function $H : P \to \mathbb{R}$, and $M \subseteq P$ a constraint submanifold. Let X be a vector field on P satisfying Eq. (34). Then, the integral curves ξ of X are solutions of the following equations,*

$$\begin{cases} \frac{\mathrm{d}q^i}{\mathrm{d}t} & = \frac{\partial H}{\partial p^i} + \lambda_a \Psi_i^a \\ \frac{\mathrm{d}p_i}{\mathrm{d}t} & = -\frac{\partial H}{\partial q^i} - p_i \frac{\partial H}{\partial z} - \lambda_a \left(\mu^a p_i + \Phi_i^a\right) \\ \frac{\mathrm{d}z}{\mathrm{d}t} & = p_i \frac{\partial H}{\partial p_i} - H + \lambda_a \left(\mu^a + p_i \Psi_i^a\right) \\ \varphi^a(\xi) & = 0, \ \forall a \end{cases} \tag{36}$$

Eq. (36) will be called *constrained Herglotz equations*. Furthermore, in case of existence and uniqueness, the solution X to Eq. (34) will be called *constrained Hamiltonian vector field* and denoted by $X_{H,M}$.

Definition 6. A *constrained Hamiltonian system* is given by a quintuple (P, η, H, M, F), where $H : P \to \mathbb{R}$ is a Hamiltonian function on a contact manifold P with contact form η, M is an embedded submanifold of P, and F is a regular distribution on P along M.

In general, the existence and uniqueness are not guaranteed.

Let be the vector subbundle of $T_M P$ defined by $\sharp(F^\circ) = {}^\perp F$, with $\sharp = \flat^{-1}$. Then, for any two solution X and Y to Eq. (34), the difference $X - Y$ is tangent to ${}^\perp F \cap TM$. In other words, we have proved the following result:

Proposition 2. *The uniqueness of solutions of Eq. (34) is equivalent to the condition*

$${}^\perp F \cap TM = \{0\}. \tag{37}$$

Let us assume that $rank(F) = dim(M)$. Then,

$${}^\perp F \oplus TM = T_M P, \tag{38}$$

where $T_M P$ consists of the tangent vectors to P at points of M. Therefore, we may consider these two projectors:

$$\mathcal{P} : T_M P \to TM, \tag{39a}$$

$$\mathcal{Q} : T_M P \to {}^\perp F. \tag{39b}$$

Let be $X = \mathcal{P}\left(X_{H|M}\right)$, where $X_{H|M}$ is the restriction to M of the Hamiltonian vector field X_H associated to H (see Eq. (15)). Then, by definition, $X \in \mathfrak{X}(M)$. On the other hand, at points of M, we have

$$\begin{aligned} &\flat(X) - \mathrm{d}H + (H + \mathcal{R}(H))\,\eta \\ &= \flat(X_H - \mathcal{Q}(X_H)) - \mathrm{d}H + (H + \mathcal{R}(H))\,\eta \\ &= -\flat(\mathcal{Q}(X_H)) \in F^\circ \end{aligned}$$

Therefore, by uniqueness, $\mathcal{P}\left(X_{H|M}\right) = X_{H,M}$ is the solution of Eq. (34).

Corollary 1. *If $rank(F) = dim(M)$, then condition (37) implies both the existence and uniqueness of solutions.*

Nevertheless, this corollary is not a necessary condition for the existence of solutions.

Lemma 5. *Eq. (34) admits solutions if, and only if,*

$$\mathrm{d}H - (H + \mathcal{R}(H))\,\eta \in \left(TM^\perp \cap F\right)^\circ$$

Proof. Notice that $\flat\left(X_{H,M}\right) \in \flat(TM) = \left(TM^\perp\right)^\circ$. Then, there exists $\Phi \in F^\circ$ such that

$$\mathrm{d}H_{|M} - (H + \mathcal{R}(H))\,\eta_{|M} = \flat\left(X_{H,M}\right) + \Phi \in \left(TM^\perp\right)^\circ + F^\circ = \left(TM^\perp \cap F\right)^\circ \quad \square$$

Furthermore, in the particular case in which the Reeb vector field $\mathcal{R}$ is tangent to the constraint submanifold (TM is vertical) or η is in F° (F is horizontal), we may improve the above result.

Corollary 2. *Assume that TM is vertical ($\mathcal{R}_{|M} \in \mathfrak{X}(M)$) or F is horizontal ($\eta \in F^\circ$). Then, Eq. (34) admits solutions if, and only if,*

$$\mathrm{d}H \in \left(TM^\perp \cap F\right)^\circ$$

Remark 3. Notice that

$$X_a = \sharp\left(\Psi^a\right) = \Psi_i^a \frac{\partial}{\partial q^i} - \left[\Phi_i^a + \mu^a p_i\right] \frac{\partial}{\partial p_i} + \left[\mu^a + \Psi_k^a p_k\right] \frac{\partial}{\partial z} \tag{40}$$

is a (local) basis of sections of the subbundle ${}^\perp F$. Then, a vector field $Y = Y^b X_b$ is tangent to M if, and only if,

$$Y^b \left[\Psi_i^b \frac{\partial \varphi^a}{\partial q^i} - \left[\Phi_i^b + \mu^b p_i\right] \frac{\partial \varphi^a}{\partial p_i} + \left[\mu^b + \Psi_k^b p_k\right] \frac{\partial \varphi^a}{\partial z}\right] = 0, \quad \forall a$$

Therefore, the matrix $\mathcal{C}$ with entries,

$$\mathcal{C}_{ab} = \Psi_i^b \frac{\partial \varphi^a}{\partial q^i} - \left[\Phi_i^b + \mu^b p_i\right] \frac{\partial \varphi^a}{\partial p_i} + \left[\mu^b + \Psi_k^b p_k\right] \frac{\partial \varphi^a}{\partial z} \tag{41}$$

represents the property (37) of uniqueness. Indeed, *the condition of uniqueness (37) is satisfied if, and only if, the rank of $\mathcal{C}$ is maximal and equal to* $2n + 1 - rank(F)$. Thus, as a consequence, a necessary condition for uniqueness of solutions of Eq. (34) is that

$$rank(F) \geq dim(M).$$

Assume now that $rank(F) = dim(M)$, i.e., Eq. (34) has the property of uniqueness and existence of solutions (Corollary 1).

Let $\{\Theta_i\}$ be a (local) basis of TM. Then, the restriction of any vector field $Y \in \mathfrak{X}(P)$ to M may be locally written as,

$$Y_{|M} = Y^i \Theta_i + \lambda^a X_a.$$

Then, applying $T\varphi^b$ we have that

$$T\varphi^b(Y) = Y\left(\varphi^b\right) = \lambda^a \mathcal{C}_{ba}.$$

Thus, we obtain

$$\lambda^a = Y\left(\varphi^b\right) \mathcal{C}^{ba}, \tag{42}$$

where $\mathcal{C}^{ba}$ are the entries of the inverse matrix of $\mathcal{C}$. Hence,

- $\mathcal{Q}\left(Y_{|M}\right) = Y\left(\varphi^b\right) \mathcal{C}^{ba} X_a$.
- $\mathcal{P}\left(Y_{|M}\right) = Y_{|M} - Y\left(\varphi^b\right) \mathcal{C}^{ba} X_a$.

Therefore, an *explicit (local) expression* of the solution $X_{H,M}$ is given by

$$X_{H,M} = (X_H)_{|M} - X_H\left(\varphi^b\right) \mathcal{C}^{ba} X_a \tag{43}$$

Notice that, for the case of $rank(F) > dim(M)$, we may still obtain the unique solution for Eq. (34) by projecting the Hamiltonian vector field X_H. In this case ${}^\perp F \oplus TM \lneq T_M P$. However, by existence condition given in Lemma 5, we have

$$X_H = \sharp\left(\mathrm{d}H_{|M}\right) - (H + \mathcal{R}(H))\,\mathcal{R} \in {}^\perp F \oplus TM.$$

Then, we may still consider the projection $X_{H,M} = \mathcal{P}\left(X_{H|M}\right)$ and it will be a solution for Eq. (34). Thus, we may calculate the solution $X_{H.M}$ to Eq. (34), by projecting the Hamiltonian vector field X_H via $\mathcal{P}$, even in *the most general case in which there is uniqueness and existence of solutions to Eq.* (34).

From now on, we will assume existence and uniqueness of solutions. Then, by applying $\flat$, we have two projections, $\overline{\mathcal{P}}$ and $\overline{\mathcal{Q}}$, of the vector subbundle, $\overline{TM} \oplus F^\circ$ of $T_M^* P$ such that $\overline{TM} = \flat(TM)$. Then, for all $a_x \in T_x^* P$, with $x \in M$, the associated projections $\overline{\mathcal{Q}} : \overline{TM} \oplus F^\circ \to F^\circ$ and $\overline{\mathcal{P}} : \overline{TM} \oplus F^\circ \to \overline{TM}$ are given by

- $\overline{\mathcal{Q}}(a_x) = \flat(\mathcal{Q}(\sharp(a_x)))$.
- $\overline{\mathcal{P}}(a_x) = \flat(\mathcal{P}(\sharp(a_x)))$.

Theorem 6. *Let X be a vector field on P. Then, X satisfies Eq. (34), if, and only if,*

$$\begin{cases} \flat(X) = \overline{\mathcal{P}}\left(\mathrm{d}H + (H + \mathcal{R}(H))\,\eta\right) \\ X_{|M} \in \mathfrak{X}(M)\,. \end{cases} \tag{44}$$

Notice that, by Lemma 5, we have that

$$\mathrm{d}H - (H + \mathcal{R}(H))\,\eta \in \left(TM^{\perp} \cap F\right)^{\circ} = \overline{TM} \oplus F^{\circ}$$

Then, it may be projected by $\overline{\mathcal{P}}$.

At this point, we would like to highlight a relevant family of constrained Hamiltonian systems.

Definition 7. A *constrained Hamiltonian system* (P, η, H, M, F) satisfies the *mechanical condition* if $\mathrm{d}H_{|M} \in {}^{\perp}F^{\circ}$.

This condition might appear "*artificial*". However, considering that $T_M P = {}^{\perp}F \oplus TM$, let $v_x \in T_M P$ such that $v_x = v_x^F + v_x^M$, where $v_x^F \in {}^{\perp}F$ and $v_x^M \in TM$. Therefore,

$$\mathrm{d}H(v_x) = \mathrm{d}H\left(v_x^M\right),$$

i.e., $\mathrm{d}H$ remains invariant for tangent vectors to the constraint manifold M, and for any other tangent vector v_x, it acts only on its component tangent to M, v_x^M. In other words,

$$\mathrm{d}H \circ \mathcal{P} = \mathrm{d}H_{|M},$$

which means that the mechanical condition can be interpreted as stating that $\mathrm{d}H_{|M}$ depends only on TM as a fiberwise linear map on $T_M P$. Thus, the derivative of the Hamiltonian at points in the constraint submanifold M depends solely on the constraint manifold.

The first consequence of this property is given in Lemma 8, which is crucial for constructing the first non-holonomic bracket.

We will demonstrate that this is, in fact, a natural condition when studying Hamiltonian and Lagrangian constrained dynamics in the subsequent sections. Specifically, we will prove that all Lagrangian and Hamiltonian systems of *mechanical type* satisfy these mechanical conditions.

Theorem 7. *Let (P, η, H, M, F) be a constrained Hamiltonian system, with $\mathcal{R}$ tangent to F. A vector field X on P satisfies*

$$\flat(X) - \mathrm{d}H + (H + \mathcal{R}(H))\,\eta \in F^{\circ}$$

if, and only if,

$$\begin{cases} \mathcal{L}_X \eta + \mathcal{R}(H)\,\eta \in F^{\circ} \\ \eta(X) = -H. \end{cases} \tag{45}$$

Proof. Assume that X fulfills the condition

$$\flat(X) - \mathrm{d}H + (H + \mathcal{R}(H))\,\eta \in F^{\circ} \tag{46}$$

Then, applying $\mathcal{R}$ ($\mathcal{R} \in F$), we have that

$$\eta(X) = -H$$

Then, the above equation reduces to

$$\iota_X \mathrm{d}\eta - \mathrm{d}\eta(X) + \mathcal{R}(H)\,\eta \in F^{\circ}$$

In other words,

$$\mathcal{L}_X \eta + \mathcal{R}(H)\,\eta \in F^{\circ}$$

The converse is proved by reversing the above arguments. □

Thus, this result permits us to give another geometric equation which is equivalent to Eq. (34). Notice the similarity with the result on energy dissipation of the Hamiltonian vector field (32). In fact, we know that contact Hamiltonian vector fields model the dynamics of dissipative systems and, contrarily to the case of symplectic Hamiltonian systems, the evolution does not preserve the energy, the contact form and the volume, i.e.,

$$\mathcal{L}_{X_H} H = -\mathcal{R}(H)H,$$
$$\mathcal{L}_{X_H} \eta = -\mathcal{R}(H)\eta,$$
$$\mathcal{L}_{X_H} \Omega = -(n+1)\mathcal{R}(H)\Omega.$$

This result may be naturally generalized to the case of nonholonomic constraints by using the projectors $\mathcal{P}$ and $\mathcal{Q}$.

Proposition 3. *Let (P, η, H, M, F) be a constrained Hamiltonian system satisfying the mechanical condition. Assume that ${}^{\perp}F \cap TM = \{0\}$ (uniqueness) and $\operatorname{rank}(F) = \dim(M)$ (existence). Then, a vector field X solving the constrained Herglotz equations (34) satisfies that*

$$\mathcal{L}_X H = -\mathcal{R}(H)\, H - \mathcal{L}_{\mathcal{Q}(X_H)} H, \tag{47a}$$
$$\mathcal{L}_X \eta = -\mathcal{R}(H)\, \eta - \mathcal{L}_{\mathcal{Q}(X_H)} \eta, \tag{47b}$$
$$\mathcal{L}_X \tilde{\eta} = -\frac{\mathcal{L}_{\mathcal{Q}(X_H)}\eta}{H}, \tag{47c}$$
$$\mathcal{L}_X \Omega = -(n+1)\mathcal{R}(H)\Omega - \eta \wedge \mathrm{d}\eta^{(n-1)} \wedge \mathrm{d}\mathcal{L}_{\mathcal{Q}(X_H)}\eta, \tag{47d}$$
$$\mathcal{L}_X \tilde{\Omega} = \tilde{\eta} \wedge \mathrm{d}\tilde{\eta}^{(n-1)} \wedge \mathrm{d}\mathcal{L}_{\mathcal{Q}(X_H)}\tilde{\eta} \tag{47e}$$

where $\tilde{\eta} = \eta/H$, assuming that H does not vanish, $\Omega = \eta \wedge (\mathrm{d}\eta)^n$ is the contact volume element and $\tilde{\Omega} = \tilde{\eta} \wedge (\mathrm{d}\tilde{\eta})^n$.

Furthermore, if $\mathcal{R} \in F$, we have that $\mathcal{L}_{\mathcal{Q}(X_H)}\eta \in F^{\circ}$.

Proof. The first item follows from $\mathcal{P}(X_H) = X$ and $\mathcal{L}_{X_H} H = -\mathcal{R}(H)\, H$. The second item can be proved in an analogous way. Then, by using Theorem 7, we have that $\mathcal{L}_{\mathcal{Q}(X_H)}\eta \in F^{\circ}$.

For the third assertion, we use the product rule

$$\begin{aligned}
\mathcal{L}_X \tilde{\eta} &= \frac{1}{H}\mathcal{L}_X \eta + X\left(\frac{1}{H}\right)\eta \\
&= -\frac{1}{H}\left(\mathcal{R}(H)\,\eta + \mathcal{L}_{\mathcal{Q}(X_H)}\eta\right) + X\left(\frac{1}{H}\right)\eta \\
&= -\frac{1}{H}\left(\mathcal{R}(H)\,\eta + \mathcal{L}_{\mathcal{Q}(X_H)}\eta\right) + \eta\frac{\mathcal{R}(H)\,H + \mathcal{L}_{\mathcal{Q}(X_H)}H}{H^2} \\
&= -\frac{\mathcal{L}_{\mathcal{Q}(X_H)}\eta}{H} + \eta\frac{\mathcal{L}_{\mathcal{Q}(X_H)}H}{H^2} \\
&= -\frac{\mathcal{L}_{\mathcal{Q}(X_H)}\eta}{H}
\end{aligned}$$

For the last equality, we have used mechanical condition. For the fourth claim, we perform the following computation.

$$\begin{aligned}
\mathcal{L}_X(\eta \wedge (\mathrm{d}\eta)^n) &= -\mathcal{R}(H)\Omega + n\eta \wedge (\mathrm{d}\eta)^{n-1} \wedge \mathrm{d}\mathcal{L}_X\eta \\
&= -(n+1)\mathcal{R}(H)\eta \wedge (\mathrm{d}\eta)^n - \eta \wedge \mathrm{d}\eta^{(n-1)} \wedge \mathrm{d}\mathcal{L}_{\mathcal{Q}(X_H)}\eta.
\end{aligned}$$

An analogous computation proves the fifth claim. □

4. Nonholonomic brackets

This section is devoted to present the nonholonomic bracket of functions, which may be defined on a constrained Hamiltonian system (P, η, H, M, F). We will present different ways to construct a nonholonomic bracket, and we study the relation between them.

First, we will construct a **nonholonomic bracket** for constrained Hamiltonian systems (P, η, H, M, F) satisfying the mechanical condition, which is a natural generalization of the one in the particular case of constrained Lagrangian contact dynamics [24]. To do this we will also assume conditions of uniqueness and existence of Proposition 2 and Corollary 1. Then, me may consider an alternative splitting of the cotangent bundle

$$T^*_M P = \left({}^{\perp}F\right)^{\circ} \oplus TM^{\circ} \tag{48}$$

where the projections are the adjoint operators $\mathcal{P}^*$ and $\mathcal{Q}^*$ of the projections $\mathcal{P}^* : T^*_M P \to \left({}^{\perp}F\right)^{\circ}$ and $\mathcal{Q}^* : T^*_M P \to TM^{\circ}$, respectively.

Thus, we can define, along M, the following geometric objects:

$$\mathcal{R}_{H,M} = \mathcal{P}\left(\mathcal{R}_{|M}\right), \tag{49}$$

$$\Lambda_{H,M} = \mathcal{P}_* \Lambda_{|M}, \tag{50}$$

where Λ is the Jacobi structure associated to the contact form η. So, for $x \in M \subseteq P$ and $\alpha, \beta \in T^*_x P$, we have

$$\Lambda_{H,M}(\alpha, \beta) = \Lambda\left(\mathcal{P}^*(\alpha), \mathcal{P}^*(\beta)\right).$$

This structure provides an associated morphism of vector bundles

$$\begin{aligned} \sharp_{\Lambda_{H,M}} : T^*_M P &\to T_M P, \\ \alpha &\mapsto \Lambda_{H,M}(\alpha, \cdot) \end{aligned} \tag{51}$$

Lemma 8. *The following identity holds*

$$\mathcal{Q}^*(\mathrm{d}H) = 0.$$

Proof. It follows directly from the mechanical condition. □

Then, by using the above Lemma, we have that,

$$\mathcal{P}(\sharp_{\Lambda}(\mathrm{d}H)) = \sharp_{\Lambda_{H,M}}(\mathrm{d}H). \tag{52}$$

Thus, we deduce

$$\begin{aligned} X_{H,M} &= \mathcal{P}(X_H) \\ &= \mathcal{P}(\sharp_{\Lambda_H}(\mathrm{d}H) - H\mathcal{R}) \\ &= \sharp_{\Lambda_{H,M}}(\mathrm{d}H) - H\mathcal{R}_{H,M}, \end{aligned}$$

along M, and, therefore, we obtain another explicit expression for the solution to Eq. (34), namely

$$X_{H,M} = \sharp_{\Lambda_{H,M}}(\mathrm{d}H) - H\mathcal{R}_{H,M}. \tag{53}$$

Furthermore, we can define the following bracket from functions on P to functions on M, which will be called the *nonholonomic bracket*:

$$\{f, g\}_{H,M} = \Lambda_{H,M}(\mathrm{d}f, \mathrm{d}g) - f\mathcal{R}_{H,M}(g) + g\mathcal{R}_{H,M}(f). \tag{54}$$

Theorem 9. *The nonholonomic bracket (54) has the following properties:*

1. *Any function g on P that is constant on M is a Casimir, i.e.,*

$$\{f, g\}_{H,M} = 0, \ \forall f \in \mathcal{C}^{\infty}\left(T^*Q \times \mathbb{R}\right)$$

2. *The bracket provides the evolution of the observables, that is,*

$$X_{H,M}(f) = \{H, f\}_{H,M} - f\mathcal{R}_{H,M}(H). \tag{55}$$

Proof. For the first assertion, let g be a function which is constant on M and let f be an arbitrary function on P. Notice that this implies that $\mathrm{d}g \in (TM)^{\circ}$, hence $\mathcal{P}^*(\mathrm{d}g) = 0$. Then, along M, we have that

$$\begin{aligned} \{g, f\}_{H,M} &= \Lambda_{H,M}(\mathrm{d}g, \mathrm{d}f) - g\mathcal{R}_{H,M}(f) + f\mathcal{R}_{H,M}(g) \\ &= \Lambda_Q(\mathcal{P}^*(\mathrm{d}g), \mathcal{P}^*(\mathrm{d}f)) = 0. \end{aligned}$$

On the other hand, by Eq. (53) we deduce,

$$\{H, f\}_{H,M} - f\mathcal{R}_{H,M}(H) = \{\sharp_{\Lambda_{H,M}}(\mathrm{d}H)\}(f) - H\mathcal{R}_{H,M}(f) = X_{H,M}(f) \quad \square$$

Notice that, in particular, all the functions φ^a defining M are Casimir's.

It is also remarkable that, using the statement *1.* in Theorem 9, the nonholonomic bracket may be restricted to functions on M. *Thus, from now on, we will refer to the nonholonomic bracket as the restriction of* $\{\cdot,\cdot\}_{H,M}$ *to functions on* M.

Proposition 4. *The nonholonomic bracket endows the space of differentiable functions on* M *with an almost Lie algebra structure which satisfies the generalized Leibniz rule*

$$\{f, gh\}_{H,M} = g\{f, h\}_{H,M} + h\{f, g\}_{H,M} - gh\mathcal{R}_{H,M}(h). \tag{56}$$

Proof. Obviously, the nonholonomic bracket is $\mathbb{R}$-linear and skew-symmetric. The generalized Leibniz rule follows from a straightforward computation. □

Corollary 3. *The vector field* $\mathcal{R}_{H,M}$ *and the bivector field* $\Lambda_{H,M}$ *induce a Jacobi structure on* M *if, and only if, the nonholonomic bracket satisfies the Jacobi identity.*

These results motivate the following definition, introduced in [24].

Definition 8. Let M be a manifold with a vector field E and a bivector field Λ. The triple (M, Λ, E) is said to be an *almost Jacobi structure* if the pair $(\mathcal{C}^\infty(M), \{\cdot,\cdot\})$ is an almost Lie algebra satisfying the generalized Leibniz rule (7) where the bracket is given by

$$\{f, g\} = \Lambda(\mathrm{d}f, \mathrm{d}g) + fE(g) - gE(f) \tag{57}$$

With this definition, we may say that the triple $\left(M, \Lambda_{H,M}, -\mathcal{R}_{H,M}\right)$ is an *almost Jacobi structure*.

We will now present two applications of this general framework. The first one retrieves the constraint dynamics for Lagrangian systems studied in [24]. The second one, permits us to give the Hamiltonian counterpart on $T^*Q \times \mathbb{R}$ from where, among other results, we will be able to present the *Hamiltonian Herglotz equations* (92).

4.1. The P-nonholonomic bracket

Notice that to construct the almost Jacobi structure $\left(M, \Lambda_{H,M}, -\mathcal{R}_{H,M}\right)$, we project the Jacobi structure induced by the contact manifold (P, η) using the projector $\mathcal{P}$. The motivation behind this construction comes from the fact that the projection of the Hamiltonian vector field X_H by $\mathcal{P}$ is the constrained Hamiltonian vector field $X_{H,M}$.

Let (P, η, H, M, F) be a constrained Hamiltonian system satisfying the *mechanical condition*, with $\mathcal{R}$ tangent to F and ${}^\perp F \subseteq F$. Then,

$$\begin{aligned}\Psi^a\left(X_{H,M}\right) &= \Psi^a\left(\sharp\left(\mathrm{d}H - (H + \mathcal{R}(H))\,\eta\right)_{|M}\right)\\ &= \Psi^a\left(\sharp(\mathrm{d}H)_{|M}\right)\\ &= 0\end{aligned}$$

The first equality is a consequence of the inclusion ${}^\perp F \subseteq F$, the second equality holds since $\mathcal{R} \in F$, and the last equality is a consequence of the mechanical condition. Then, *the mechanical condition implies that the constrained Hamiltonian vector field* $X_{H,M}$ *is tangent to* F*, i.e.,* $X_{H,M} \in F$.

Let us now assume the conditions of uniqueness in Proposition 2 and existence in Corollary 1. Then, we have that

$${}^\perp TM \cap F = \{0\}. \tag{58}$$

Therefore, $TM \cap F$ satisfies that

$$(TM \cap F) \oplus {}^\perp(TM \cap F) = T_M P \tag{59}$$

So, we obtain the projections,

$$\mathrm{P} : T_M P \to TM \cap F \tag{60a}$$

$$\mathrm{Q} : T_M P \to {}^\perp(TM \cap F) \tag{60b}$$

Notice that $\mathcal{P}\left(X_{H|M}\right) = X_{H,M} \in TM \cap F$ and $\mathcal{Q}\left(X_{H|M}\right) \in {}^\perp F \subseteq {}^\perp(TM \cap F)$. Thus, by the uniqueness, we get

$$\mathrm{P}\left(X_{H|M}\right) = \mathcal{P}\left(X_{H|M}\right) = X_{H,M}$$

Therefore, P *provides another projection to obtain the constrained Hamiltonian vector field associated to the constrained Hamiltonian system* (P,η,H,M,F) satisfying the mechanical condition in such a way that the Reeb vector field $\mathcal{R}$ is tangent to F and ${}^{\perp}F\subseteq F$.

Consider a (local) basis of 1-forms $\{\Psi^a=\Phi_i^a\mathrm{d}q^i+\Psi_i^a\mathrm{d}p_i+\mu^a\mathrm{d}z\}$ of F°. Then, for all a,b

$$\Psi^b\left(\sharp\left(\Psi^a\right)\right)=\left(\Phi_i^b\Psi_i^a-\Phi_i^a\Psi_i^b\right)+\left(\Psi_i^a\mu^b p_i-\Psi_i^b\mu^a p_i\right)+\mu^a\mu^b$$

So, ${}^{\perp}F\subseteq F$ and $\mathcal{R}\in F$ are satisfied if, and only if, we have that

$$\begin{aligned}\mu^a&=0\\ \Phi_i^b\Psi_i^a-\Phi_i^a\Psi_i^b&=0,\end{aligned}$$

for all a,b.

Let φ^b be constraint functions locally defining M. Then, ${}^{\perp}TM$ is generated by

$$\begin{aligned}Y_c&=\sharp\left(\mathrm{d}\varphi^c\right)\\&=\frac{\partial\varphi^c}{\partial q^i}\sharp\left(\mathrm{d}q^i\right)+\frac{\partial\varphi^c}{\partial p_i}\sharp\left(\mathrm{d}p_i\right)+\frac{\partial\varphi^c}{\partial z}\sharp\left(\mathrm{d}z\right)\\&=-\frac{\partial\varphi^c}{\partial q^i}\frac{\partial}{\partial p_i}+\frac{\partial\varphi^c}{\partial p_i}\left(\frac{\partial}{\partial q_i}+p_i\frac{\partial}{\partial z}\right)+\frac{\partial\varphi^c}{\partial z}\left(\frac{\partial}{\partial z}-p_i\frac{\partial}{\partial p_i}\right)\\&=\frac{\partial\varphi^c}{\partial p_i}\frac{\partial}{\partial q_i}-\left(\frac{\partial\varphi^c}{\partial q^i}+p_i\frac{\partial\varphi^c}{\partial z}\right)\frac{\partial}{\partial p_i}+\left(\frac{\partial\varphi^c}{\partial z}+p_i\frac{\partial\varphi^c}{\partial p_i}\right)\frac{\partial}{\partial z}\end{aligned}$$

So, ${}^{\perp}(TM\cap F)$ is locally generated by the family of (local) vector fields $\{Y_c,X_a\}$, where,

$$X_a=\sharp\left(\Psi^a\right)=\Psi_i^a\frac{\partial}{\partial q^i}-\left[\Phi_i^a+\mu^a p_i\right]\frac{\partial}{\partial p_i}+\left[\mu^a+\Psi_k^a p_k\right]\frac{\partial}{\partial z}$$

is a (local) basis of sections of the subbundle ${}^{\perp}F$. Consider a (local) basis $\{\theta_i\}$ of $TM\cap F$. Then, the restriction of any (local) vector field $Y\in\mathfrak{X}(P)$ to M may be locally written as,

$$Y_{|M}=R^i\theta_i+\nu^cY_c+\lambda^aX_a.$$

Then, by applying Ψ^d (${}^{\perp}F\subseteq F$), we get

$$\Psi^d(Y)=\nu^c\Psi^d(Y_c)$$

Let us denote by $\mathcal{G}$ the matrix with entries,

$$\mathcal{G}_{dc}=\Psi^d(Y_c).$$

Hence,

$$\nu^c=\Psi^d(Y)\,\mathcal{G}^{dc},$$

where $\mathcal{G}^{dc}$ are the entries of the inverse matrix of $\mathcal{G}$. On the other hand, applying $\mathrm{d}\varphi^b$, we have that

$$\mathrm{d}\varphi^b(Y)=Y\left(\varphi^b\right)=\nu^cY_c\left(\varphi^b\right)+\lambda^a\mathcal{C}_{ba},$$

where $\mathcal{C}$ is the matrix with entries given by Eq. (41). Therefore,

$$\lambda^a=\left[Y\left(\varphi^b\right)-\nu^cY_c\left(\varphi^b\right)\right]\mathcal{C}^{ba}=\left[Y\left(\varphi^b\right)-\Psi^d(Y)\,\mathcal{G}^{dc}Y_c\left(\varphi^b\right)\right]\mathcal{C}^{ba},$$

where $\mathcal{C}^{ba}$ are the entries of the inverse matrix of $\mathcal{C}$. Hence,

$$\begin{aligned}\mathrm{Q}\left(Y_{|M}\right)&=\Psi^d(Y)\,\mathcal{G}^{dc}Y_c+\left[Y\left(\varphi^b\right)-\Psi^d(Y)\,\mathcal{G}^{dc}Y_c\left(\varphi^b\right)\right]\mathcal{C}^{ba}X_a.\\ \mathrm{P}\left(Y_{|M}\right)&=Y_{|M}-\Psi^d(Y)\,\mathcal{G}^{dc}Y_c-\left[Y\left(\varphi^b\right)-\Psi^d(Y)\,\mathcal{G}^{dc}sY_c\left(\varphi^b\right)\right]\mathcal{C}^{ba}X_a.\end{aligned}$$

Therefore, an *explicit (local) expression* of the solution $X_{H,M}$ is given by

$$X_{H,M}=(X_H)_{|M}-\Psi^d(X_H)\,\mathcal{G}^{dc}Y_c-\left[X_H\left(\varphi^b\right)-\Psi^d(Y)\,\mathcal{G}^{dc}Y_c\left(\varphi^b\right)\right]\mathcal{C}^{ba}X_a \tag{61}$$

Thus, we have another local expression for the constrained Hamiltonian vector field $X_{H,M}$.

So, it makes sense to define the following almost Jacobi structure,

$$\overline{\mathcal{R}}_{H,M} = \mathrm{P}\left(\mathcal{R}_{|M}\right), \tag{62}$$

$$\overline{\Lambda}_{H,M} = \mathrm{P}_{*}\Lambda_{|M}. \tag{63}$$

Then, we may define the following bracket of functions on M

$$\{f,g\}^{\mathrm{P}}_{H,M} = \overline{\Lambda}_{H,M}(\mathrm{d}f,\mathrm{d}g) - f\overline{\mathcal{R}}_{H,M}(g) + g\overline{\mathcal{R}}_{H,M}(f). \tag{64}$$

Let us call this bracket as P-*nonholonomic bracket*.

Notice that, as a consequence of ${}^{\perp}F \subseteq F$, we may conclude that any vector Y tangent to F satisfies that

$$\mathcal{P}(Y) \;=\; \mathrm{P}(Y)$$

Therefore, $\overline{\mathcal{R}}_{H,M} = \mathcal{R}_{H,M}$.

On the other hand, we may consider the adjoint operators

$$\mathcal{P}^{*} : T^{*}_{M}P \to \left({}^{\perp}F\right)^{\circ}$$
$$\mathcal{Q}^{*} : T^{*}_{M}P \to TM^{\circ}$$
$$\mathrm{P}^{*} : T^{*}_{M}P \to \left({}^{\perp}(F\cap TM)\right)^{\circ}$$
$$\mathrm{Q}^{*} : T^{*}_{M}P \to (F\cap TM)^{\circ}$$

Then, applying $\sharp$ to ${}^{\perp}F^{\circ}$ and TM°, we obtain a new decomposition of $T^{*}_{M}P$,

$$T^{*}_{M}P = F \oplus {}^{\perp}TM$$

with associated projections, $\mathbb{P} : T^{*}_{M}P \to F$ and $\mathbb{Q} : T^{*}_{M}P \to {}^{\perp}TM$ given by

$$\mathbb{P} = \sharp \circ \mathcal{P}^{*} \circ \flat, \qquad \mathbb{Q} = \sharp \circ \mathcal{Q}^{*} \circ \flat$$

Performing the same operation with the projections P^{*} and Q^{*}, we obtain the splitting

$$T_{M}P = (TM\cap F) \oplus {}^{\perp}(TM\cap F)$$

Hence, by uniqueness, we deduce that

$$\sharp \circ \mathrm{P}^{*} \circ \flat = \mathrm{P}, \qquad \sharp \circ \mathrm{Q}^{*} \circ \flat = \mathrm{Q} \tag{65}$$

Let us now consider $V, W \in T_{M}P$. Then,

$$\begin{aligned}\Lambda_{H,M}(\flat(V),\flat(W)) &= \Lambda\left(\mathcal{P}^{*}(\flat(V)),\mathcal{P}^{*}(\flat(W))\right)\\ &= -\mathrm{d}\eta\left(\sharp\left(\mathcal{P}^{*}(\flat(V))\right),\sharp\left(\mathcal{P}^{*}(\flat(W))\right)\right)\\ &= -\mathrm{d}\eta\left(\mathbb{P}(V),\mathbb{P}(W)\right)\end{aligned}$$

Notice that, $\mathcal{R}\in F$ implies that $\eta\left({}^{\perp}F\right) = 0$, and therefore,

$$\begin{aligned}\mathrm{d}\eta_{\mathrm{Q}}\left(\mathrm{Q}(\mathbb{P}(V)),\mathbb{P}(W)\right) &= \left\{\flat_{\mathrm{Q}}\left(\mathrm{Q}(\mathbb{P}(V))\right)\right\}(\mathbb{P}(W))\\ &= 0\end{aligned}$$

The last equality follows since $\mathbb{P}(W)\in F$ and $\flat\left(\mathrm{Q}\left(\mathbb{P}(V)\right)\right)\in F$. Hence,

$$\begin{aligned}\Lambda_{H,M}(\flat(V),\flat(W)) &= -\mathrm{d}\eta\left(\mathrm{P}(\mathbb{P}(V)),\mathbb{P}(W)\right)\\ &= -\mathrm{d}\eta\left(\mathrm{P}(\mathbb{P}(V)),\mathrm{P}(\mathbb{P}(W))\right)\end{aligned}$$

Observe that

$$V = \mathrm{P}(\mathbb{P}(V)) + \mathrm{Q}(\mathbb{P}(V)) + \mathbb{Q}(V)$$

Nevertheless, $\mathrm{Q}\left(\mathbb{P}(V)\right)\in {}^{\perp}F \subseteq F$ and $\mathbb{Q}(V)\in {}^{\perp}TM$. Then,

$$\mathrm{Q}(\mathbb{P}(V)) + \mathbb{Q}(V) \in {}^{\perp}[F\cap TM]$$

Therefore, we get

$$\mathrm{P}\left(\mathbb{P}\left(V\right)\right)=\mathrm{P}\left(V\right)$$

In other words, we obtain

$$\begin{aligned}\Lambda_{H,\mathcal{M}}\left(\flat\left(V\right),\flat\left(W\right)\right)&=-\mathrm{d}\eta\left(\mathrm{P}\left(\mathbb{P}\left(V\right)\right),\mathrm{P}\left(\mathbb{P}\left(W\right)\right)\right)\\&=-\mathrm{d}\eta\left(\mathrm{P}\left(V\right),\mathrm{P}\left(W\right)\right)\\&=\overline{\Lambda}_{H,\mathcal{M}}\left(\flat\left(V\right),\flat\left(W\right)\right)\end{aligned}$$

Thus, we have proved the following theorem.

Theorem 10. *The nonholonomic bracket* $\{\cdot,\cdot\}_{H,M}$ *and the* P*-nonholonomic bracket* $\{\cdot,\cdot\}^{\mathrm{P}}_{H,M}$ *coincide.*

Notice that both the Lagrangian and the Hamiltonian descriptions satisfy the mechanical condition, the Reeb vector field is tangent to F, and ${}^{\perp}F\subseteq F$ (see the next two sections for a proof). Therefore, this equivalent version of the nonholonomic bracket may be defined in both cases, giving a *completely new way* (in both cases) of calculating the bracket of function in a (Hamiltonian or Lagrangian) nonholonomic contact system.

5. Contact Lagrangian dynamics

A particular but relevant case of contact manifold is given by the manifold $TQ\times\mathbb{R}$ where Q is an n-dimensional manifold, equipped with a *regular Lagrangian function* $L:TQ\times\mathbb{R}\to\mathbb{R}$, i.e., a function on $TQ\times\mathbb{R}$ satisfying that the matrix,

$$W_{ij}=\left(\frac{\partial^2 L}{\partial\dot{q}^i\partial\dot{q}^j}\right) \tag{66}$$

is regular. So, we may construct a contact form on $TQ\times\mathbb{R}$ as follows

$$\eta_L=\mathrm{d}z-\alpha_L, \tag{67}$$

where $\alpha_L=S^*(\mathrm{d}L)$, and S is the *canonical vertical endomorphism on* TQ extended in the natural way to $TQ\times\mathbb{R}$ (see [21]). η_L is called the *contact Lagrangian form*. Let $\left(q^i,\dot{q}^i,z\right)$ the natural coordinates on $TQ\times\mathbb{R}$. Then,

$$\eta_L=\mathrm{d}z-\frac{\partial L}{\partial\dot{q}^i}\mathrm{d}q^i. \tag{68}$$

Moreover, the energy of the system is defined as follows

$$E_L=\Delta(L)-L=\dot{q}^i\frac{\partial L}{\partial\dot{q}^i}-L, \tag{69}$$

where $\Delta=\dot{q}^i\dfrac{\partial}{\partial\dot{q}^i}$ is the *Liouville vector field on* TQ extended naturally to $TQ\times\mathbb{R}$. The triple $\left(TQ\times\mathbb{R},\eta_L,E_L\right)$ is called a *contact Lagrangian system*.

The dynamics is given by the *Euler–Lagrangian vector field*, which is $\Gamma_L=X_{E_L}$ (Eq. (15)) whose integral curves $\left(\xi,\dot{\xi},s\right)$ are solutions of the following equations

$$\frac{\partial L}{\partial q^i}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}^i}+\frac{\partial L}{\partial\dot{q}^i}\frac{\partial L}{\partial z}=0 \tag{70}$$

$$\dot{z}=L \tag{71}$$

These equations were presented by G. Herglotz in 1930 [31] (see also [29,28]), and they will be called *Herglotz equations*. In a more recent approach, A. Bravetti et al. [9] and M. de León and M. Lainz [17] have obtained the connection between these equations and the contact Hamiltonian dynamics. Furthermore, we may derive these Herglotz equations from the *Herglotz variational principle* [17, Thm. 2].

In [24], the nonholonomic equations for contact Lagrangian systems with linear kinematic constraints were studied, by restricting the variations so that they satisfy the nonholonomic constraints in the Herglotz variational principle. We will prove that this may be understood as a particular case of our general framework of constrained dynamics.

Let us assume that we have linear kinematic constraints (that is, constraints linear on the velocities) defined by a regular distribution $\mathcal{D}$ on the configuration manifold Q with codimension k. Then, there exists a family of local 1-forms $\{\Phi^a\}_{a=1,\dots,k}$ on Q such that

$$\mathcal{D} = \left\{ v \in TQ \mid \Phi^a(v) = 0 \right\}. \tag{72}$$

Consider the Lagrangian function $L : TQ \times \mathbb{R} \to \mathbb{R}$, and define the set

$$\Omega(q_1, q_2, [a,b])^{\mathcal{D}}_{\xi} = \{ v \in T_\xi \Omega(q_1, q_2, [a,b]) \mid v(t) \in \mathcal{D}_{\xi(t)} \text{ for all } t \in [a,b] \}, \tag{73}$$

where, $\Omega(q_1, q_2, [a,b]) \subseteq (\mathcal{C}^\infty([a,b] \to Q)$ is the infinite dimensional manifold given by the smooth curves ξ on an interval $[a,b] \subset \mathbb{R}$ such that $\xi(a) = q_1$ and $\xi(b) = q_2$. Then,

$$T_\xi \Omega(q_1, q_2, [a,b]) = \{ v_\xi \in \mathcal{C}^\infty([a,b] \to TQ) \mid \tau_Q \circ v_\xi = \xi,\ v_\xi(a) = 0,\ v_\xi(b) = 0 \}.$$

So, ξ satisfies the *Herglotz variational principle with constraints* [24] if, and only if,

1. $T_\xi \mathcal{A}_{|\Omega(q_1,q_2,[a,b])^{\mathcal{D}}_{\xi}} = 0$.
2. $\dot{\xi}(t) \in \mathcal{D}_{\xi(t)}$ for all $t \in [a,b]$.

Theorem 11 *([24]). A path $\xi \in \Omega(q_1, q_2, [a,b])$ satisfies the Herglotz variational principle with constraints if and only if*

$$\begin{cases} \left(\dfrac{\partial L}{\partial q^i} - \dfrac{\mathrm{d}}{\mathrm{d}t} \dfrac{\partial L}{\partial \dot{q}^i} + \dfrac{\partial L}{\partial \dot{q}^i} \dfrac{\partial L}{\partial z} \right) dq^i \in \mathcal{D}^\circ_{\xi(t)} \\ \dot{\xi} \in \mathcal{D} \end{cases} \tag{74}$$

where $\mathcal{D}^\circ = \{ a \in T^*Q \mid a(u) = 0 \text{ for all } u \in \mathcal{D} \}$ is the annihilator of $\mathcal{D}$.

In other words, there exist some Lagrange multipliers $\lambda_i(q^i)$ such that,

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} \dfrac{\partial L}{\partial \dot{q}^i} - \dfrac{\partial L}{\partial q^i} - \dfrac{\partial L}{\partial \dot{q}^i} \dfrac{\partial L}{\partial z} = \lambda_a \Phi^a_i \\ \Phi^a(\dot{\xi}(t)) = 0. \end{cases} \tag{75}$$

From now on, Eqs. (75) will be called the *constrained Herglotz equations*.

The distribution $\mathcal{D}$ may be naturally lifted into a distribution $\mathcal{D}^l$ on $TQ \times \mathbb{R}$ in such a way that

$$(\mathcal{D}^l)^\circ = \left(\tau_{Q,\mathbb{R}}\right)^* \mathcal{D}^\circ = S^* \left(T(\mathcal{D} \times \mathbb{R})^\circ \right) = < \tilde{\Phi}^a = \Phi^a_i \mathrm{d}q^i >, \tag{76}$$

where $\tau_{Q,\mathbb{R}} : TQ \times \mathbb{R} \to Q$ is the projection given by $\tau_{Q,\mathbb{R}}(v_q, z) = q$.

Theorem 12 *([24]). Assume that L is regular. Let X be a vector field on $TQ \times \mathbb{R}$ satisfying the equation*

$$\begin{cases} \flat_L(X) - \mathrm{d}E_L + (E_L + \mathcal{R}_L(E_L))\, \eta_L \in \mathcal{D}^{l\circ} \\ X_{|\mathcal{D}\times\mathbb{R}} \in \mathfrak{X}(\mathcal{D} \times \mathbb{R}). \end{cases} \tag{77}$$

Then,

(1) *X is a SODE on $TQ \times \mathbb{R}$.*
(2) *The integral curves of X are solutions of the constrained Herglotz equations (75).*

Remark 4. Recall that a *SODE* is a vector field X on $TQ \times \mathbb{R}$ such that,

$$S(X) = \Delta \tag{78}$$

Let (q^i) be a local chart on Q. Then, Eq. (78) reduces to

$$X\left(q^i\right) = \dot{q}^i,\ \forall i.$$

Thus, X is a SODE if and only if any integral curve of X is written (locally) as $(\xi, \dot{\xi}, s)$ for some (local) curve ξ on Q. Observe that, it is immediate to prove that the vector field X defined above is a SODE.

By considering $M = \mathcal{D} \times \mathbb{R}$ and $F = \mathcal{D}^l$, we have that Eq. (77) reduces to Eq. (34) and the quintuple $\left(TQ \times \mathbb{R}, \eta_L, E_L, \mathcal{D} \times \mathbb{R}, \mathcal{D}^l\right)$ is a constrained Hamiltonian system satisfying the *mechanical condition* (Definition 7).

Indeed, the local basis of F° is given by the 1-forms $\tilde{\Phi}^a = \Phi_i^a \mathrm{d}q^i$, and it is trivial to check the mechanical condition and ${}^\perp F \subseteq F$. Furthermore, $\mathcal{R}_L\left(\tilde{\Phi}^a\right) = 0$ for all a, i.e., F is vertical.

Furthermore, as it is proved in [24], the condition of existence in Corollary 1 is always satisfied, and the condition of uniqueness of Proposition 2 is satisfied, if and only if, the matrix,

$$(\mathcal{C}_{ab}) = \left(W^{ik}\Phi_k^b \Phi_i^a\right), \tag{79}$$

where $\left(W^{ik}\right)$ is the inverse of the Hessian matrix $(W_{ik}) = \left(\dfrac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^k}\right)$, is regular.

It is a simple exercise to check that the *nonholonomic bracket* for the constrained Hamiltonian system $\left(TQ \times \mathbb{R}, \eta_L, E_L, \mathcal{D} \times \mathbb{R}, \mathcal{D}^l\right)$ given in Eq. (54), coincides with the one given in [24].

6. Constrained dynamics of contact Hamiltonian systems

In this section, we will use the general framework of constrained dynamics developed in Section 3, to obtain the Hamiltonian counterpart of nonholonomic Lagrangian systems.

Here, we will consider a *Hamiltonian function* as a differentiable function $H : T^*Q \times \mathbb{R} \to \mathbb{R}$. We may define the canonical contact form of $T^*Q \times \mathbb{R}$ given by

$$\eta_Q = dz - \theta_Q, \tag{80}$$

where θ_Q is the *canonical Liouville form on* $T^*Q \times \mathbb{R}$. Namely,

$$\{\theta_Q\left(\alpha_q, z\right)\}\left(X_{\alpha_q}, \lambda_z\right) = \alpha_q\left(T_{\alpha_q}\pi_{Q,\mathbb{R}}\left(X_{\alpha_q}\right)\right)$$

for all $\left(\alpha_q, z\right) \in T^*Q \times \mathbb{R}$, and $\left(X_{\alpha_q}, \lambda_z\right) \in T_{(\alpha_q, z)}\left(T^*Q \times \mathbb{R}\right)$, where $\pi_{Q,\mathbb{R}} : T^*Q \times \mathbb{R} \to Q$ is the projection defined by $\pi_{Q,\mathbb{R}}\left(\alpha_q, z\right) = q$.

Thus, considering the canonical coordinates $\left(q^i, p_i, z\right)$, we have that

$$\eta_Q = dz - p_i \mathrm{d}q^i. \tag{81}$$

Then, the Hamiltonian vector associated to H is locally described as follows,

$$X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial H}{\partial q^i} + p_i \frac{\partial H}{\partial z}\right)\frac{\partial}{\partial p_i} + \left(p_i \frac{\partial H}{\partial p_i} - H\right)\frac{\partial}{\partial z} \tag{82}$$

Therefore, its integral curves satisfy the *contact Hamiltonian equations*

$$\begin{cases} \dfrac{\mathrm{d}q^i}{\mathrm{d}t} &= \dfrac{\partial H}{\partial p^i} \\ \dfrac{\mathrm{d}p_i}{\mathrm{d}t} &= -\dfrac{\partial H}{\partial q^i} - p_i \dfrac{\partial H}{\partial z} \\ \dfrac{\mathrm{d}z}{\mathrm{d}t} &= p_i \dfrac{\partial H}{\partial p_i} - H \end{cases} \tag{83}$$

From a physical intuitive view, we will derive the Hamiltonian constrained equations from the Lagrangian constrained equations via the Legendre transformation. The *Legendre transformation*, is defined as follows,

$$\begin{array}{rccl} FL: & TQ \times \mathbb{R} & \to & T^*Q \times \mathbb{R} \\ & \left(v_q, z\right) & \mapsto & \mathrm{d}\left(L_{T_qQ \times \{z\}}\right)_{|(v_q, z)} \end{array}$$

i.e., locally

$$FL\left(q^i, \dot{q}^i, z\right) = \left(q^i, \frac{\partial L}{\partial \dot{q}^i}, z\right)$$

Along this paper, we will assume that FL is a global diffeomorphism, namely, L is *hyperregular*. Notice that, in this case, the *Hamiltonian energy* is defined by

$$H = E_L \circ (FL)^{-1} \tag{84}$$

Then,

$$\begin{cases} FL^*\eta_Q & = \eta_L \\ FL_*\mathcal{R}_L & = \mathcal{R}_Q \\ FL_*\Gamma_L & = X_H \\ T^*FL \circ \flat_Q \circ TFL & = \flat_L \end{cases} \tag{85}$$

Hence, FL transforms the Herglotz equations (70) into the contact Hamiltonian equations (83).

Let us now assume the existence of linear kinematic constraints (on the velocities) defined by a regular distribution $\mathcal{D}$ on the configuration manifold Q with codimension k and a family of local 1-forms $\{\Phi^a\}_{a=1,\dots,k}$ on Q generating $\mathcal{D}^\circ$ (see Eq. (72)). Let us denote by M the image of the constraint manifold $\mathcal{D} \times \mathbb{R}$ by the Legendre transformation FL, say $M = FL(\mathcal{D} \times \mathbb{R})$.

Assume that there exists a solution $\Gamma_{L,\mathcal{D}}$ of the Eq. (77). Then, we may define a vector field $X_{H,M} = FL_*(\Gamma_{L,\mathcal{D}})$ on $T^*Q \times \mathbb{R}$. So, by taking into account Eqs. (85), $X_{H,M}$ satisfies,

$$\begin{cases} FL^*\left[\flat_Q\left(X_{H,M}\right) - \mathrm{d}H + \left(H + \mathcal{R}_Q\left(H\right)\right)\eta_Q\right] \in \mathcal{D}^{l^\circ} \\ \left(X_{H,M}\right)_{|M} \in \mathfrak{X}\left(M\right). \end{cases} \tag{86}$$

Notice that, by Eq. (76), we have

$$FL^{-1^*}\left(\mathcal{D}^{l^\circ}\right) = FL^{-1^*}\left(\left(\tau_{Q,\mathbb{R}}\right)^*\mathcal{D}^\circ\right) \tag{87}$$

$$= \left(\tau_{Q,\mathbb{R}} \circ FL^{-1}\right)^*\mathcal{D}^\circ \tag{88}$$

$$= \left(\pi_{Q,\mathbb{R}}\right)^*\mathcal{D}^\circ \tag{89}$$

We will denote by $\mathfrak{D}^l$ the vector subbundle of $T^*Q \times \mathbb{R}$ such that $\left(\pi_{Q,\mathbb{R}}\right)^*\mathcal{D}^\circ = \mathfrak{D}^{l^\circ}$. Then, the annihilator $\mathfrak{D}^{l^\circ}$ is also generated by the 1-forms $\Phi_i^a dq^i$ on $T^*Q \times \mathbb{R}$ (see Eq. (72)) and, we may rewrite Eq. (86) as follows

$$\begin{cases} \flat_Q\left(X_{H,M}\right) - \mathrm{d}H + \left(H + \mathcal{R}_Q\left(H\right)\right)\eta_Q \in \mathfrak{D}^{l^\circ} \\ X_{|M} \in \mathfrak{X}\left(M\right). \end{cases} \tag{90}$$

So, for the Hamiltonian counterpart we have the constrained Hamiltonian system given by $\left(T^*Q \times \mathbb{R}, \eta_Q, H, FL(\mathcal{D} \times \mathbb{R}), \mathfrak{D}^l\right)$ (see Definition 6). Therefore, locally, the constrained Hamiltonian vector field is given by,

$$X_{H,M} = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial H}{\partial q^P i} + p_i\frac{\partial H}{\partial z} + \lambda_a\Phi_i^a\right)\frac{\partial}{\partial p_i} + \left(p_i\frac{\partial H}{\partial p_i} - H\right)\frac{\partial}{\partial z} \tag{91}$$

and its integral curves ξ satisfy the following equations,

$$\begin{cases} \frac{\mathrm{d}q^i}{\mathrm{d}t} & = \frac{\partial H}{\partial p^i} \\ \frac{\mathrm{d}p_i}{\mathrm{d}t} & = -\frac{\partial H}{\partial q^i} - p_i\frac{\partial H}{\partial z} - \lambda_a\Phi_i^a \\ \frac{\mathrm{d}z}{\mathrm{d}t} & = p_i\frac{\partial H}{\partial p_i} - H \\ \varphi^a\left(\xi\right) & = 0, \ \forall a \end{cases} \tag{92}$$

where φ^a are the constraint functions defined by $\Phi_a = \phi_a \circ FL$. From now on, Eqs. (92) will be called the *constrained Hamiltonian Herglotz equations*. Thus, we have proved the following result.

Theorem 13. *Let be a Hamiltonian function $H : T^*Q \times \mathbb{R} \to \mathbb{R}$, and a constraint manifold $M \subseteq T^*Q \times \mathbb{R}$. Let X be a vector field on $T^*Q \times \mathbb{R}$ satisfying the equation*

$$\begin{cases} \flat_Q\left(X\right) - \mathrm{d}H + \left(H + \mathcal{R}_Q\left(H\right)\right)\eta_Q \in \mathfrak{D}^{l^\circ} \\ X_{|M} \in \mathfrak{X}\left(M\right) \end{cases}$$

Then, the integral curves of X are solutions of the constrained Hamiltonian Herglotz equations (92). A solution for this equation will be denoted by $X_{H,M}$, and will be called constrained Hamiltonian vector field.

Notice that, in general, FL preserves the fibers of $\tau_{Q,\mathbb{R}}$ and $\pi_{Q,\mathbb{R}}$, but it is not linear on these fibers. Thus, we may not assume that M is "*linear*", i.e., a subbundle of $\pi_{Q,\mathbb{R}}$. Indeed, the momentum coordinates of the Legendre transformation FL depend on the real variable "z", so we may not even assume that $M = \mathcal{M} \times \mathbb{R}$, for some subset $\mathcal{M}$ of T^*Q.

Nevertheless, there is a relevant family of Lagrangian functions which transform the space $\mathcal{D}\times\mathbb{R}$, with $\mathcal{D}$ a regular distribution of Q, into the space $\mathcal{M}\times\mathbb{R}$, with $\mathcal{M}$ a subbundle of T^*Q; these are those of *mechanical type*. A Lagrangian L is of *mechanical type* if there exists a Riemannian metric g on Q, such that,

$$L\left(v_q,z\right) = \frac{1}{2}g\left(v_q,v_q\right)+V\left(q,z\right),\ \forall\left(v_q,z\right)\in T_qQ\times\mathbb{R}.$$

Here, $T\left(v_q\right)=\frac{1}{2}g\left(v_q,v_q\right)$ represent the *kinetic energy*, and V is the *potential energy*. Then,

$$FL\left(v_q,z\right)=\left(\flat_g\left(v_q\right),z\right),$$

where $\flat_g$ is the *musical isomorphism* associated to g. Thus, in this case, we have

$$FL\left(\mathcal{D}\times\mathbb{R}\right)=\flat_g\left(\mathcal{D}\right)\times\mathbb{R},$$

and $\mathcal{M}=\flat_g\left(\mathcal{D}\right)=\mathcal{D}^{\perp_g\circ}$ is a vector subbundle of T^*Q isomorphic to $\mathcal{D}$, where $\cdot^{\perp_g}$ defines the orthogonal complement respect to the Riemannian metric g. We will denote the inverse of $\flat_g$ by $\sharp_g$, i.e., $\flat_g^{-1}=\sharp_g$.

Remark 5. For an arbitrary vector subbundle $\mathcal{M}$ of T^*Q, we may define the annihilator

$$\mathcal{M}^\circ := \{v\in TQ \mid \alpha\left(v\right)=0,\ \forall\alpha\in\mathcal{M}\}$$

Then, $\mathcal{M}^\circ$ is generated by local vector fields Z_a. So, in the Hamiltonian side the local vector fields Z_a will play the role of the *constraints*. Indeed,

$$\flat_g\left(\mathcal{M}^\circ\right) = \left(\sharp_g\left(\mathcal{M}\right)\right)^\circ$$

which means that the 1-forms,

$$\flat_g\left(Z_a\right) = \Phi^a,$$

generates $\mathcal{D}^\circ=\left(\sharp_g\left(\mathcal{M}\right)\right)^\circ$. Notice that any vector field X on Q may be equivalently depicted as a function $X:T^*Q\to\mathbb{R}$ which is linear on the fibers of T^*Q. So, the *linear constraints* of the Hamiltonian system will be given by the vector fields Z_a, as maps from T^*Q to $\mathbb{R}$.

Therefore, we may rewrite the constrained Hamiltonian equations (92) in terms of a subbundle $\mathcal{M}$ of T^*Q of codimension k. Notice that, using Eq. (84), we have that our Hamiltonian function is given by

$$H\left(\alpha_q,z\right)=T\left(\sharp_g\left(\alpha_q\right),\sharp_g\left(\alpha_q\right)\right)-V\left(q,z\right)=\frac{1}{2}\left(\alpha_q\left(\sharp_g\left(\alpha_q\right)\right)\right)-V\left(q,z\right) \tag{93}$$

Locally,

$$H\left(q^i,p_i,z\right)=\frac{1}{2}g^{i,j}p_ip_j-V\left(q^i,z\right), \tag{94}$$

where $g^{i,j}$ are the entries of the inverse matrix of $\left(g_{i,j}=g\left(\frac{\partial}{\partial q^i},\frac{\partial}{\partial q^j}\right)\right)$. In fact, we may prove the following result for the case in which the constraints are given by a subbundle $\mathcal{M}$ of T^*Q.

Theorem 14. *Let be a Hamiltonian function $H:T^*Q\times\mathbb{R}\to\mathbb{R}$, and a constraint manifold $\mathcal{M}\subseteq T^*Q$ which is a subbundle of T^*Q. Let X be a vector field on $T^*Q\times\mathbb{R}$ satisfying the equations*

$$\begin{cases}\flat_Q\left(X\right)-\mathrm{d}H+\left(H+\mathcal{R}_Q\left(H\right)\right)\eta_Q\in\mathfrak{D}^{l^\circ}\\ X_{|\mathcal{M}\times\mathbb{R}}\in\mathfrak{X}\left(\mathcal{M}\times\mathbb{R}\right),\end{cases} \tag{95}$$

where

$$\mathfrak{D}^{l^\circ}=\left(\pi_{Q,\mathbb{R}}\right)^*\flat_g\left(\mathcal{M}^\circ\right)$$

Then, the integral curves of X are solutions of the following constrained Hamiltonian Herglotz equations,

$$\begin{cases}\frac{\mathrm{d}q^i}{\mathrm{d}t} & =\frac{\partial H}{\partial p^i}\\ \frac{\mathrm{d}p_i}{\mathrm{d}t} & =-\frac{\partial H}{\partial q^i}-p_i\frac{\partial H}{\partial z}-\lambda_a\Phi_i^a\\ \frac{\mathrm{d}z}{\mathrm{d}t} & =p_i\frac{\partial H}{\partial p_i}-H\\ \varphi^a\left(\flat_g\left(\dot{\xi}\right)\right) & =0,\ \forall a\end{cases} \tag{96}$$

In this case, the constrained Hamiltonian vector field will be denoted by $X_{H,\mathcal{M}}$. Notice that

$$\mathfrak{D}^{l^\circ} = \left(\pi_{Q,\mathbb{R}}\right)^* \flat_g \left(\mathcal{M}^\circ\right) = \left(\pi_{Q,\mathbb{R}}\right)^* \left(\sharp_g \left(\mathcal{M}\right)\right)^\circ = \left(\pi_{Q,\mathbb{R}}\right)^* \flat_g \left(\sharp_g \left(\mathcal{M}\right)^{\perp_g}\right) \tag{97}$$

Therefore, Eq. (92) (resp. Eq. (96)) provides the correct nonholonomic dynamics in the Hamiltonian side. From now on, we will restrict to the case in which the constraints are given by a vector subbundle $\mathcal{M}$ of T^*Q (see Theorem 14). In other words, we will restrict to the constrained Hamiltonian systems given by the quintuples $\left(T^*Q \times \mathbb{R}, \eta_Q, H, \mathcal{M} \times \mathbb{R}, \mathfrak{D}^l\right)$, where $\mathcal{M}$ is a subbundle of the cotangent bundle T^*Q.

Let us now investigate the existence and uniqueness of solutions via Proposition 2 and Corollary 1.

From the coordinate expression of the constraints $\tilde{\Phi}^a = \Phi_i^a \mathrm{d}q^i$ defining $\mathfrak{D}^l$, one can see that $\mathcal{R}_Q\left(\tilde{\Phi}^a\right) = 0$, hence $\mathfrak{D}^l$ is vertical in the sense of Definition 4. Thus, it satisfies the condition *i*) of Definition 7.

Let $\{Z_a\}_{a=1\ldots k}$ be a local basis of $\mathcal{M}^\circ$, and

$$\flat_g \left(Z_a^j \frac{\partial}{\partial q^j}\right) = \left[Z_a^j g_{j,i}\right] \mathrm{d}q^i,$$

where $g_{i,j} = g\left(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j}\right)$ are the local components of g. Then, $\tilde{\Phi}^a = \left[Z_a^j g_{j,i}\right] \mathrm{d}q^i$ is a local basis of $\mathfrak{D}^{l^\circ}$. Therefore,

$$X_a = \sharp_Q \left(\tilde{\Phi}^a\right) = -\left[Z_a^j g_{j,i}\right] \frac{\partial}{\partial p_i} \tag{98}$$

is a (local) basis of sections of the subbundle ${}^\perp\mathfrak{D}^l$. Observe that, it is trivial to prove that

$${}^\perp\mathfrak{D}^l \subseteq \mathfrak{D}^l \tag{99}$$

Theorem 15. *The Hamiltonian Herglotz equations (95) have the property of existence and uniqueness of solutions.*

Proof. Notice that $T\left(\mathcal{M} \times \mathbb{R}\right)$ may be considered as a subbundle of $TQ \times \mathbb{R}$ along the submanifold $\mathcal{M} \times \mathbb{R}$. Then, it is easy to show that the annihilator of the distribution $T\left(\mathcal{M} \times \mathbb{R}\right)$ is given by $\mathrm{pr}^*_{\mathcal{M}\times\mathbb{R}}\left(T\mathcal{M}\right)^\circ$ where $\mathrm{pr}_{\mathcal{M}\times\mathbb{R}} : \mathcal{M} \times \mathbb{R} \to \mathcal{M}$ denotes the projection onto the first component. Then, let $Y = Y^b X_b$ be a vector field on $\mathcal{M} \times \mathbb{R}$ tangent to ${}^\perp\mathfrak{D}^l$. Hence, we have that

$$Y^b T Z_a \left(X_b\right) = 0.$$

Equivalently,

$$Y^b Z_b^j g_{j,i} Z_a^i = 0, \ \forall a.$$

Then, $Y^b = 0$, for all b.

On the other hand, for each $\left(\alpha_q, z\right) \in \mathcal{M} \times \mathbb{R}$ we have that

- $\dim\left({}^\perp\mathfrak{D}^l{}_{|(\alpha_q,z)}\right) = k$
- $\dim\left(T_{(\alpha_q,z)}\left(\mathcal{M} \times \mathbb{R}\right)\right) = 2n + 1 - k$

So, uniqueness implies existence. In fact,

$${}^\perp\mathfrak{D}^l \oplus T\left(\mathcal{M} \times \mathbb{R}\right) = T_{\mathcal{M}\times\mathbb{R}}\left(T^*Q \times \mathbb{R}\right), \tag{100}$$

where $T_{\mathcal{M}\times\mathbb{R}}\left(T^*Q \times \mathbb{R}\right)$ consists of the tangent vectors to $T^*Q \times \mathbb{R}$ at points in $\mathcal{M} \times \mathbb{R}$. □

Thus, we may consider these two projectors

$$\mathcal{P} : T_{\mathcal{M}\times\mathbb{R}}\left(T^*Q \times \mathbb{R}\right) \to T\left(\mathcal{M} \times \mathbb{R}\right), \tag{101a}$$

$$\mathcal{Q} : T_{\mathcal{M}\times\mathbb{R}}\left(T^*Q \times \mathbb{R}\right) \to {}^\perp\mathfrak{D}^l, \tag{101b}$$

and the Hamiltonian constrained vector field $X_{H,\mathcal{M}}$ is obtained by projecting the Hamiltonian vector field X_H with $\mathcal{P}$, i.e., $\mathcal{P}\left(X_{H|\mathcal{M}\times\mathbb{R}}\right) = X_{H,\mathcal{M}}$ is the solution of Eq. (95). Furthermore, an explicit (local) expression of the solution $X_{H,\mathcal{M}}$ is given by

$$X_{H,\mathcal{M}} = \left(X_H\right)_{|\mathcal{M}\times\mathbb{R}} - X_H\left(Z_b\right) \mathcal{C}^{ba} X_a \tag{102}$$

Recall that, by applying $\flat_Q$, we have a splitting on the cotangent bundle,

$$T^*_{\mathcal{M}\times\mathbb{R}}\left(T^*Q\times\mathbb{R}\right)=\overline{T\left(\mathcal{M}\times\mathbb{R}\right)}\oplus\mathfrak{D}^{l^\circ},$$

in such a way that $\overline{T\left(\mathcal{M}\times\mathbb{R}\right)}=\flat_Q\left(T\left(\mathcal{M}\times\mathbb{R}\right)\right)$ and $T^*_{\mathcal{M}\times\mathbb{R}}\left(T^*Q\times\mathbb{R}\right)$ are the 1-forms on $T^*Q\times\mathbb{R}$ at points of $\mathcal{M}\times\mathbb{R}$. Then, for all $a_{(\alpha_q,z)}\in T^*_{(\alpha_q,z)}\left(T^*Q\times\mathbb{R}\right)$ with $(\alpha_q,z)\in\mathcal{M}\times\mathbb{R}$, the associated projections $\overline{\mathcal{Q}}:T^*_{\mathcal{M}\times\mathbb{R}}\left(T^*Q\times\mathbb{R}\right)\to\mathfrak{D}^{l^\circ}$ and $\overline{\mathcal{P}}:T^*_{\mathcal{M}\times\mathbb{R}}\left(T^*Q\times\mathbb{R}\right)\to\overline{T\left(\mathcal{M}\times\mathbb{R}\right)}$ are given by

- $\overline{\mathcal{Q}}\left(a_{(\alpha_q,z)}\right)=\flat_Q\left(\mathcal{Q}\left(\sharp_Q\left(a_{(\alpha_q,z)}\right)\right)\right)$.
- $\overline{\mathcal{P}}\left(a_{(\alpha_q,z)}\right)=\flat_Q\left(\mathcal{P}\left(\sharp_Q\left(a_{(\alpha_q,z)}\right)\right)\right)$.

Furthermore, we have the following results, as corollaries of the corresponding ones in Section 3.

Theorem 16. *Let X be a vector field on $T^*Q\times\mathbb{R}$. Then X satisfies Eq. (95), if, and only if,*

$$\begin{cases}\flat_Q\left(X\right)=\overline{\mathcal{P}}\left(\mathrm{d}H+\left(H+\mathcal{R}_Q\left(H\right)\right)\eta_Q\right)\\ X_{|\mathcal{M}\times\mathbb{R}}\in\mathfrak{X}\left(\mathcal{M}\times\mathbb{R}\right).\end{cases}\tag{103}$$

Theorem 17. *A vector field X on $T^*Q\times\mathbb{R}$ satisfies*

$$\flat_Q\left(X\right)-\mathrm{d}H+\left(H+\mathcal{R}_Q\left(H\right)\right)\eta_Q\in\mathfrak{D}^{l^\circ}$$

if and only if

$$\begin{cases}\mathcal{L}_X\eta_Q+\mathcal{R}_Q\left(H\right)\eta_Q\in\mathfrak{D}^{l^\circ}\\ \eta_Q\left(X\right)=-H.\end{cases}\tag{104}$$

Notice that, by Eq. (99), we have that

$$\begin{aligned}\tilde{\Phi}_a\left(X_{H,\mathcal{M}}\right)&=\tilde{\Phi}_a\left(X_{H|\mathcal{M}\times\mathbb{R}}\right)\\&=\tilde{\Phi}_a\left(\sharp_Q\left(\mathrm{d}H-\left(H+\mathcal{R}_Q\left(H\right)\right)\eta_Q\right)_{|\mathcal{M}\times\mathbb{R}}\right)\\&=\tilde{\Phi}_a\left(\sharp_Q\left(\mathrm{d}H\right)_{|\mathcal{M}\times\mathbb{R}}\right)\end{aligned}$$

Then, $X_{H,\mathcal{M}}-\sharp_Q\left(\mathrm{d}H\right)_{|\mathcal{M}\times\mathbb{R}}\in\mathfrak{D}^l$. Hence, $X_{H,\mathcal{M}}\in\mathfrak{D}^l$ if, and only if, $\sharp_Q\left(\mathrm{d}H\right)_{|\mathcal{M}\times\mathbb{R}}\in\mathfrak{D}^l$. On the other hand, for all $(\alpha_q,z)\in\mathcal{M}\times\mathbb{R}$, we have

$$\begin{aligned}\{\tilde{\Phi}_a\left(\sharp_Q\left(\mathrm{d}H\right)_{|\mathcal{M}\times\mathbb{R}}\right)\}\left(\alpha_q,z\right)&=\{\Phi_a^i\frac{\partial H}{\partial p_i}\}\left(\alpha_q,z\right)\\&=\{\Phi_a^i g^{i,k}p_k\}\left(\alpha_q,z\right)\\&=\Phi_a\left(\sharp_g\left(\alpha_q\right)\right)\\&=0\end{aligned}$$

Therefore,

$$X_{H,\mathcal{M}}\in\mathfrak{D}^l\quad,\quad\mathrm{d}H_{|\mathcal{M}\times\mathbb{R}}\in\flat\left(\mathfrak{D}^l\right)=\mathfrak{D}^{l^{\perp^\circ}}\tag{105}$$

Namely, it is satisfied the mechanical condition (see Definition 7). In other words, we have proved that $\left(T^*Q\times\mathbb{R},\eta_Q,H,\mathcal{M}\times\mathbb{R},\mathfrak{D}^l\right)$ satisfies the *mechanical condition* in Definition 7, as well as, ${}^\perp\mathfrak{D}^l\subseteq\mathfrak{D}^l$, $\mathcal{R}_Q\in\mathfrak{D}^l$, and conditions of existence and uniqueness of the solutions in Proposition 2 and Corollary 1.

Notice that, the mechanical condition also implies that

$$T\pi_{Q,\mathbb{R}}\left(X_{H,\mathcal{M}}\right)\in\mathcal{D}.$$

As a consequence of satisfying these conditions, we still may consider another projection on $T_{\mathcal{M}\times\mathbb{R}}\left(T^*Q\times\mathbb{R}\right)$ by the splitting,

$$T\left(\mathcal{M}\times\mathbb{R}\right)\cap\mathfrak{D}^l\oplus\left[T\left(\mathcal{M}\times\mathbb{R}\right)\cap\mathfrak{D}^l\right]^{\perp}=T_{\mathcal{M}\times\mathbb{R}}\left(T^*Q\times\mathbb{R}\right)$$

with the projections,

$$\mathrm{P}:T_{\mathcal{M}\times\mathbb{R}}\left(T^*Q\times\mathbb{R}\right)\to T\left(\mathcal{M}\times\mathbb{R}\right)\cap\mathfrak{D}^l, \tag{106a}$$

$$\mathrm{Q}:T_{\mathcal{M}\times\mathbb{R}}\left(T^*Q\times\mathbb{R}\right)\to\left[T\left(\mathcal{M}\times\mathbb{R}\right)\cap\mathfrak{D}^l\right]^{\perp}. \tag{106b}$$

Therefore,

$$\mathrm{P}\left(X_{H|\mathcal{M}\times\mathbb{R}}\right)=\mathcal{P}\left(X_{H|\mathcal{M}\times\mathbb{R}}\right)=X_{H,\mathcal{M}}$$

Observe that, in this case, it also happens that $\mathcal{R}_Q\in T\left(\mathcal{M}\times\mathbb{R}\right)$. Then,

$${}^{\perp}\left[T\left(\mathcal{M}\times\mathbb{R}\right)\cap\mathfrak{D}^l\right]=\left[T\left(\mathcal{M}\times\mathbb{R}\right)\cap\mathfrak{D}^l\right]^{\perp}.$$

Finally, we may define a *nonholonomic bracket* for the constrained Hamiltonian system $\left(H,T^*Q\times\mathbb{R},\mathcal{M}\times\mathbb{R},\mathfrak{D}^l\right)$ given in Eq. (54).

Remark 6. Recall that, in this case, the vector fields Z_a, as fiberwise linear map over $T^*Q\times\mathbb{R}$, define locally $\mathcal{M}\times\mathbb{R}$, and

$$X_a=\sharp_Q\left(\tilde{\Phi}^a\right)=-\left[Z_a^j g_{j,i}\right]\frac{\partial}{\partial p_i}$$

is a (local) basis of sections of the subbundle ${}^{\perp}\mathfrak{D}^l$. Then, a basis of ${}^{\perp}TM$, is given by

$$Y_c=Z_c^i\frac{\partial}{\partial q_i}-p_l\frac{\partial Z_c^l}{\partial q^i}\frac{\partial}{\partial p_i}+p_iZ_c^i\frac{\partial}{\partial z}$$

Therefore, the matrix $\mathcal{G}$ has the following entries

$$\mathcal{G}_{dc}=\tilde{\Phi}^d\left(Y_c\right)=Z_d^j g_{j,i}Z_c^i=X_d\left(Z_c\right)=-\mathcal{C}_{dc} \tag{107}$$

So, ${}^{\perp}(TM\cap F)$ is locally generated by the family of (local) vector fields $\{Y_c,X_a\}$, where

$$X_a=\sharp\left(\Psi^a\right)=\Psi_i^a\frac{\partial}{\partial q^i}-\left[\Phi_i^a+\mu^a p_i\right]\frac{\partial}{\partial p_i}+\left[\mu^a+\Psi_k^a p_k\right]\frac{\partial}{\partial z}$$

is a (local) basis of sections of the subbundle ${}^{\perp}F$. Consider a (local) basis $\{\theta_i\}$ of $TM\cap F$. Then, the restriction of any (local) vector field $Y\in\mathfrak{X}(P)$ to M may be locally written as

$$Y_{|M}=R^i\theta_i+\nu^cY_c+\lambda^aX_a.$$

Then, by applying Ψ^d (${}^{\perp}F\subseteq F$), we get

$$\Psi^d\left(Y\right)=\nu^c\Psi^d\left(Y_c\right)$$

Let us denote by $\mathcal{G}$ the matrix with entries,

$$\mathcal{G}_{dc}=\Psi^d\left(Y_c\right).$$

Hence,

$$\nu^c=\Psi^d\left(Y\right)\mathcal{G}^{dc},$$

where $\mathcal{G}^{dc}$ are the entries of the inverse matrix of $\mathcal{G}$. On the other hand, applying $\mathrm{d}\varphi^b$ we have that

$$\mathrm{d}\varphi^b\left(Y\right)=Y\left(\varphi^b\right)=\nu^cY_c\left(\varphi^b\right)+\lambda^a\mathcal{C}_{ba},$$

where $\mathcal{C}$ is the matrix with entries given by Eq. (41). Therefore,

$$\lambda^a=\left[Y\left(\varphi^b\right)-\nu^cY_c\left(\varphi^b\right)\right]\mathcal{C}^{ba},$$

where $\mathcal{C}^{ba}$ are the entries of the inverse matrix of $\mathcal{C}$. Hence,

$$\mathrm{Q}\left(Y_{|M}\right)=\Psi^d\left(Y\right)\mathcal{G}^{dc}Y_c+\left[Y\left(\varphi^b\right)-\Psi^d\left(Y\right)\mathcal{G}^{dc}Y_c\left(\varphi^b\right)\right]\mathcal{C}^{ba}X_a.$$

$$\mathrm{P}\left(Y_{|M}\right)=Y_{|M}-\Psi^d\left(Y\right)\mathcal{G}^{dc}Y_c-\left[Y\left(\varphi^b\right)-\Psi^d\left(Y\right)\mathcal{G}^{dc}sY_c\left(\varphi^b\right)\right]\mathcal{C}^{ba}X_a.$$

Notice that, in this case, the Reeb vector field $\mathcal{R}_Q$ is also tangent to $\mathcal{M}\times\mathbb{R}$. Then, the nonholonomic bracket is defined along $\mathcal{M}\times\mathbb{R}$ as follows

$$\{f,g\}_{H,\mathcal{M}} = \Lambda_{H,\mathcal{M}}(\mathrm{d}f,\mathrm{d}g) - f\mathcal{R}_Q(g) + g\mathcal{R}_Q(f). \tag{108}$$

where

$$\Lambda_{H,\mathcal{M}} = \mathcal{P}_*\Lambda_{Q\,|\mathcal{M}\times\mathbb{R}},$$

Λ_Q being the Jacobi structure associated to the canonical contact form η_Q. Then, along $\mathcal{M}\times\mathbb{R}$, we have

$$X_{H,\mathcal{M}} = \sharp_{\Lambda_{H,\mathcal{M}}}(\mathrm{d}H) - H\mathcal{R}_Q.$$

Recall that Theorem 9 shows us that this bracket may be restricted to $\mathcal{M}\times\mathbb{R}$, and provides the correct evolution of the observables by the formula

$$X_{H,\mathcal{M}}(f) = \{H,f\}_{H,\mathcal{M}} - f\mathcal{R}_Q(H). \tag{109}$$

Let us describe locally this bracket. Let $\left(q^i\right)$ be a local system of coordinates on Q. Then,

- $\mathcal{P}^*\left(\mathrm{d}q^i\right) = \mathrm{d}q^i$
- $\mathcal{P}^*\left(\mathrm{d}p_i\right) = \mathrm{d}p_i + g_{m,i}Z_a^m\mathcal{C}^{ba}p_l\frac{\partial Z_b^l}{\partial q^j}\,\mathrm{d}q^j + Z_b^j g_{l,i}Z_a^l\mathcal{C}^{ba}\mathrm{d}p_j$
- $\mathcal{P}^*\left(\mathrm{d}z\right) = \mathrm{d}z$

This implies that

- $\sharp_Q\left(\mathcal{P}^*\left(\mathrm{d}q^i\right)\right) = -\frac{\partial}{\partial p_i}$
- $\sharp_Q\left(\mathcal{P}^*\left(\mathrm{d}p_i\right)\right) = \frac{\partial}{\partial q_i} + \left[Z_b^j g_{l,i}Z_a^l\mathcal{C}^{ba} - g_{m,i}Z_a^m\mathcal{C}^{ba}p_l\frac{\partial Z_b^l}{\partial q^j}\right]\frac{\partial}{\partial q_j} + \left[p_i + Z_b^j g_{l,i}Z_a^l\mathcal{C}^{ba}p_j\right]\frac{\partial}{\partial z}$
- $\sharp_Q\left(\mathcal{P}^*\left(\mathrm{d}z\right)\right) = p_i\frac{\partial}{\partial p_i} + \frac{\partial}{\partial z}$

So,

- $\Lambda_{H,\mathcal{M}}\left(\mathrm{d}q^i,\mathrm{d}q^j\right) = \Lambda_{H,\mathcal{M}}\left(\mathrm{d}z,\mathrm{d}z\right) = \Lambda_{H,\mathcal{M}}\left(\mathrm{d}q^i,\mathrm{d}z\right) = \Lambda_{H,\mathcal{M}}\left(\mathrm{d}p_i,\mathrm{d}p_j\right) = 0.$
- $\Lambda_{H,\mathcal{M}}\left(\mathrm{d}p_i,\mathrm{d}q^j\right) = \delta_i^j + Z_b^j g_{l,i}Z_a^l\mathcal{C}^{ba} - g_{m,i}Z_a^m\mathcal{C}^{ba}p_l\frac{\partial Z_b^l}{\partial q^j}$
- $\Lambda_{H,\mathcal{M}}\left(\mathrm{d}p_i,\mathrm{d}z\right) = p_i + p_r\left(Z_b^r g_{l,i}Z_a^l\mathcal{C}^{ba} - g_{m,i}Z_a^m\mathcal{C}^{ba}p_l\frac{\partial Z_b^l}{\partial q^r}\right)$

Therefore, by taking into account that $\mathcal{R}_Q = \frac{\partial}{\partial z}$, we have that the local expression of the nonholonomic bracket is given by

- $\{q^i,q^j\}_{H,\mathcal{M}} = \{p^i,p^j\}_{H,\mathcal{M}} = \{z,z\}_{H,\mathcal{M}} = 0.$
- $\{p_i,q^j\}_{H,\mathcal{M}} = \delta_i^j + Z_b^j g_{l,i}Z_a^l\mathcal{C}^{ba} - g_{m,i}Z_a^m\mathcal{C}^{ba}p_l\frac{\partial Z_b^l}{\partial q^j}$
- $\{p_i,z\}_{H,\mathcal{M}} = p_r\left(Z_b^r g_{l,i}Z_a^l\mathcal{C}^{ba} - g_{m,i}Z_a^m\mathcal{C}^{ba}p_l\frac{\partial Z_b^l}{\partial q^r}\right)$
- $\{q^i,z\}_{H,\mathcal{M}} = -q^i$

6.1. The contact Eden bracket

We can also define another nonholonomic bracket, based in the projection given by the Riemannian metric defining the Lagrangian function.

Notice that,

$$TQ = \mathcal{D}\oplus\mathcal{D}^{\perp_g},$$

where $\mathcal{D} = \sharp_Q(\mathcal{M})$. Then, by applying $\flat_g$, we have that

$$T^*Q = \mathcal{M}\oplus\mathcal{D}^\circ$$

In particular,

$$T^*Q\times\mathbb{R} = (\mathcal{M}\times\mathbb{R})\oplus\mathcal{D}^\circ \tag{110}$$

Let us consider the projection $\gamma : T^*Q \times \mathbb{R} \to \mathcal{M} \times \mathbb{R}$. Then, it turns natural to define the following bracket of functions,

$$\{f, g\}_E = \{f \circ \gamma, g \circ \gamma\}_{|\mathcal{M}\times\mathbb{R}} \tag{111}$$

for all $f, g \in \mathcal{C}^\infty\left(\mathcal{M} \times \mathbb{R}\right)$. This bracket will be called **contact Eden bracket** following the spirit of [26,27].

Let be $Z \in T_{(\alpha_q,s)}\left(T^*Q \times \mathbb{R}\right)$, $\alpha_q \in \mathcal{M}$. Consider a curve $(\alpha(t), f(t)) \subseteq T^*Q \times \mathbb{R}$ in such a way that $\left(\dot{\alpha}(0), \dot{f}(0)\right) = Z$, $\alpha(0) = \alpha_q$ and $f(0) = s$. So, we may define

$$\Lambda(t) = (\alpha(t), f(t)) - \gamma((\alpha(t), f(t))) \in \mathcal{D}^\circ_{b(t)},$$

where $b(t) = \pi_{Q,\mathbb{R}}(\alpha(t), f(t)) = \pi_{Q,\mathbb{R}}(\gamma(\alpha(t), f(t)))$. Notice that γ leaves the coordinate on $\mathbb{R}$ invariant.

Then,

$$\Lambda(t) = \lambda_a(t)\,\Phi^a(b(t)) = \lambda_a(t)\,\Phi^a_i(b(t))\,dq^i_{|b(t)}. \tag{112}$$

Observe that $\Lambda(0) = 0_q \in T^*_qQ$, which implies that $\lambda_a(0) = 0$, for all a (since $\{\Phi^a\}$ is a basis of $\mathcal{D}^\circ$). Therefore, by Eq. (112), we have in coordinates

$$\begin{aligned}\dot{\Lambda}(0) &= \dot{b}(0)\frac{\partial}{\partial q_i}_{|0_q} + \left[\dot{\alpha}^i(0) - \gamma\,\dot{(\alpha)}^i(0)\right]\frac{\partial}{\partial p_i}_{|0_q}\\ &= \dot{b}(0)\frac{\partial}{\partial q_i}_{|0_q} + \dot{\lambda}_a(0)\,\Phi^a_i\left(\alpha_q\right)\frac{\partial}{\partial p_i}_{|0_q}\end{aligned}$$

Notice that we have used that $\lambda_a(0) = 0$, for all a.

On the other hand, observe that

$$\begin{aligned}Z &= \dot{b}(0)\frac{\partial}{\partial q_i}_{|(\alpha_q,s)} + \dot{\alpha}^i(0)\frac{\partial}{\partial p_i}_{|(\alpha_q,s)}\\ T\gamma(Z) &= \dot{b}(0)\frac{\partial}{\partial q_i}_{|(\alpha_q,s)} + \gamma\,\dot{(\alpha)}^i(0)\frac{\partial}{\partial p_i}_{|(\alpha_q,s)}\end{aligned}$$

where $\alpha(t) = \alpha^i(t)\,\frac{\partial}{\partial q_i}_{|b(t)}$, and $\gamma(\alpha)(t) = \gamma(\alpha)^i(t)\,\frac{\partial}{\partial q_i}_{|b(t)}$. Thus,

$$\begin{aligned}X = Z - T\gamma(Z) &= \left[\dot{\alpha}^i(t) - \gamma\,\dot{(\alpha)}^i(t)\right]\frac{\partial}{\partial p_i}_{|(\alpha_q,s)}\\ &= \dot{\lambda}_a(0)\,\Phi^a_i\left(\alpha_q\right)\frac{\partial}{\partial p_i}_{|(\alpha_q,s)}\end{aligned}$$

Observe that $\eta_Q(X) = 0$. Therefore,

$$\flat_Q(X) = \iota_X \mathrm{d}\eta_Q = -X(p_i)\,dq^i_{|(\alpha_q,s)} = -\dot{\lambda}_a(0)\,\Phi^a_i\left(\alpha_q\right)dq^i_{|(\alpha_q,s)}$$

In other words, $\flat_Q(X) \in \mathfrak{D}^{l\circ}$ or, equivalently, $X \in {}^{\perp}\mathfrak{D}^l$. Thus, we will prove the following result

Theorem 18. *Let be $\alpha_q \in \mathcal{M}$. Then,*

$$T\gamma(Z) = \mathcal{P}(Z),$$

for all $Z \in T_{(\alpha_q,s)}\left(T^*Q \times \mathbb{R}\right)$.

Proof. Take into account that $Z = T\gamma(Z) + X$, with $X = Z - T\gamma(Z) \in {}^{\perp}\mathfrak{D}^l$. Furthermore, obviously $T\gamma(Z) \in T\left(\mathcal{M} \times \mathbb{R}\right)$. Then, by uniqueness, we obtain the result. □

Notice that, as an immediate corollary, we deduce

Corollary 4. $T\gamma\left(X_{H|\mathcal{M}\times\mathbb{R}}\right) = X_{H,\mathcal{M}}$.

Recall that the adjoint operator of $\mathcal{P}$ is given by, $\mathcal{P}^* : T^*_{\mathcal{M}\times\mathbb{R}}\left(T^*Q\times\mathbb{R}\right) \to {}^{\perp}\mathfrak{D}^{l^\circ}$, such that, for all $\alpha \in T^*_{(\alpha_q,s)}\left(T^*Q\times\mathbb{R}\right)$ and $Z \in T_{(\alpha_q,s)}\left(T^*Q\times\mathbb{R}\right)$, we have

$$\begin{aligned}\mathcal{P}^*\alpha\left(Z\right) &= \alpha\left(\mathcal{P}\left(Z\right)\right)\\ &= \alpha\left(T\gamma\left(Z\right)\right)\\ &= \left\{T^*\gamma\left(\alpha\right)\right\}\left(Z\right)\end{aligned}$$

Hence, $T^*\gamma = \mathcal{P}^*$.

Theorem 19. *The contact Eden bracket $\{\cdot,\cdot\}_E$ and the nonholonomic bracket $\{\cdot,\cdot\}_{H,\mathcal{M}}$ coincide, namely*

$$\{f,g\}_E = \{f\circ\gamma, g\circ\gamma\}_{|\mathcal{M}\times\mathbb{R}}$$

Proof. Recall that the nonholonomic bracket was defined on functions over $\mathcal{M}\times\mathbb{R}$ as follows

$$\{f,g\}_{H,\mathcal{M}} = \Lambda_{H,\mathcal{M}}(\mathrm{d}f,\mathrm{d}g) - f\mathcal{R}_{H,\mathcal{M}}(g) + g\mathcal{R}_{H,\mathcal{M}}(f),$$

where

$$\begin{aligned}\mathcal{R}_{H,\mathcal{M}} &= \mathcal{P}\left(\mathcal{R}_{Q\,|\mathcal{M}\times\mathbb{R}}\right),\\ \Lambda_{H,\mathcal{M}} &= \mathcal{P}_*\Lambda_{Q\,|\mathcal{M}\times\mathbb{R}},\end{aligned}$$

with Λ_Q as the Jacobi structure associated to the contact form η_Q. So, for $f, g \in \mathcal{M}\times\mathbb{R}$, we have that

$$\begin{aligned}\Lambda_{H,\mathcal{M}}\left(\mathrm{d}f,\mathrm{d}g\right) &= \Lambda_{Q\,|\mathcal{M}\times\mathbb{R}}\left(\mathcal{P}^*\left(\mathrm{d}f\right),\mathcal{P}^*\left(\mathrm{d}g\right)\right)\\ &= \Lambda_{Q\,|\mathcal{M}\times\mathbb{R}}\left(T^*\gamma\left(\mathrm{d}f\right),T^*\gamma\left(\mathrm{d}g\right)\right)\\ &= \Lambda_{Q\,|\mathcal{M}\times\mathbb{R}}\left(\mathrm{d}\left(f\circ\gamma\right),\mathrm{d}\left(g\circ\gamma\right)\right)\end{aligned}$$

On the other hand,

$$\mathcal{R}_{H,\mathcal{M}}\left(f\right) = \mathcal{P}\left(\mathcal{R}_{Q\,|\mathcal{M}\times\mathbb{R}}\right)\left(f\right) = \left\{T\gamma\left(\mathcal{R}_{Q\,|\mathcal{M}\times\mathbb{R}}\right)\right\}\left(f\right) = \mathcal{R}_{Q\,|\mathcal{M}\times\mathbb{R}}\left(f\circ\gamma\right)$$

Therefore,

$$\begin{aligned}\{f,g\}_{H,\mathcal{M}} &= \Lambda_{H,\mathcal{M}}(\mathrm{d}f,\mathrm{d}g) - f\mathcal{R}_{H,\mathcal{M}}(g) + g\mathcal{R}_{H,\mathcal{M}}(f)\\ &= \Lambda_{Q\,|\mathcal{M}\times\mathbb{R}}\left(\mathrm{d}\left(f\circ\gamma\right),\mathrm{d}\left(g\circ\gamma\right)\right)\\ &\quad - f\mathcal{R}_{Q\,|\mathcal{M}\times\mathbb{R}}\left(g\circ\gamma\right) + g\mathcal{R}_{Q\,|\mathcal{M}\times\mathbb{R}}\left(f\circ\gamma\right)\\ &= \{f\circ\gamma, g\circ\gamma\}_{|\mathcal{M}\times\mathbb{R}}\end{aligned}$$

Notice that, the last equality is a consequence of this fact $(f\circ\gamma)_{|\mathcal{M}\times\mathbb{R}} = f$. So,

$$\{f,g\}_E = \{f\circ\gamma, g\circ\gamma\}_{|\mathcal{M}\times\mathbb{R}} \quad \square$$

At this point, it is important to note that we have restricted our results to a Hamiltonian constrained system induced by a Lagrangian of mechanical type. However, we actually do not need the existence of a Riemannian metric to construct the Eden bracket, and to prove the associated results. In fact, it is enough to consider a constrained Hamiltonian system given by the quintuple $\left(T^*Q\times\mathbb{R},\eta_Q,H,\mathcal{M}\times\mathbb{R},\mathfrak{D}^l\right)$, where $\mathcal{M}$ is a subbundle of the cotangent bundle T^*Q in such a way that

$$T^*Q = \mathcal{M}\oplus\mathcal{D}^\circ$$

So, in general, we may easily generalize the Eden bracket, for the case in which our constrained Hamiltonian comes from a splitting of the cotangent bundle,

$$T^*Q = \mathcal{M}\oplus\mathcal{N}$$

where $\mathcal{M}$ and $\mathcal{N}$ are vector subbundles of T^*Q. Then, the constrained Hamiltonian system is defined by the quintuples $\left(T^*Q\times\mathbb{R},\eta_Q,H,\mathcal{M}\times\mathbb{R},\mathfrak{N}^l\right)$, where

$$\left(\pi_{Q,\mathbb{R}}\right)^*\mathcal{N} = \mathfrak{N}^{l^\circ}$$

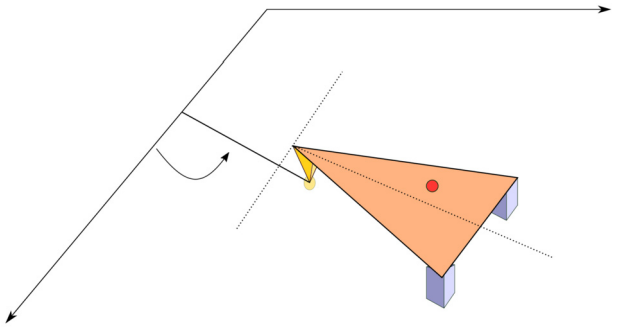


Fig. 1. Chaplygin's sleigh.

7. Chaplygin's sleigh

In this section, we work with a model for a sleigh with homogeneous and isotropic Rayleigh dissipation.

A comprehensive study of Chaplygin's sleigh can be found in [12,37]. The nonholonomic nature of this system has been analyzed in [20].

The system described in [20,33] has a configuration space given by $Q = \mathbb{R}^2 \times S^1$, with (x, y) representing the coordinates of the center of mass of the sleigh and an angle θ describing its orientation. We introduce an additional term, γz, which accounts for friction with the medium following the Rayleigh dissipation model [39,9,8]. Consequently, the resulting Lagrangian is given by,

$$L = \frac{1}{2}\left[\dot{x}^2 + \dot{y}^2 + J\dot{\phi}^2\right] + \gamma z \tag{113}$$

where γ is the friction coefficient of the carriage with the medium, J is the moment of inertia and we are assuming that the mass of the sleight m is equal to 1 (see Fig. 1). The units are normalized so that the mass and the radius of inertia of the sleight is 1. We also denote by a the distance from the contact point of the sleigh and the center of mass.

Thus, L is of mechanical type and the associated Riemannian metric is given by,

$$g = \mathrm{d}x^2 + \mathrm{d}y^2 + J\mathrm{d}\phi^2$$

The linear kinematic constraints are defined by the following regular distribution,

$$\mathcal{D} = \left\langle \frac{\partial}{\partial x} - \frac{\sin\phi}{a}\frac{\partial}{\partial \phi}, \frac{\partial}{\partial y} + \frac{\cos\phi}{a}\frac{\partial}{\partial \phi} \right\rangle$$

Then,

$$\mathcal{D}^\circ = \left\langle \mathrm{d}\phi - \frac{\cos\phi}{a}\mathrm{d}y + \frac{\sin\phi}{a}\mathrm{d}x \right\rangle$$

In other words, the linear constraints are given by the equation:

$$\dot{\phi} - \frac{\cos\phi}{a}\dot{y} + \frac{\sin\phi}{a}\dot{x} = 0.$$

Therefore, the orthogonal complement is given by,

$$\mathcal{D}^{\perp_g} = \left\langle J\frac{\sin\phi}{a}\frac{\partial}{\partial x} - J\frac{\cos\phi}{a}\frac{\partial}{\partial y} + \frac{\partial}{\partial \phi} \right\rangle$$

Now, we consider,

$$\begin{aligned}\mathcal{M} &= \flat_g(\mathcal{D}) \\ &= \left\langle \mathrm{d}x - J\frac{\sin\phi}{a}\mathrm{d}\phi, \mathrm{d}y + J\frac{\cos\phi}{a}\mathrm{d}\phi \right\rangle\end{aligned}$$

Let us denote the canonical coordinates of the cotangent bundle of Q by $\left((x, y, \phi), \overline{x}, \overline{y}, \overline{\phi}\right)$. Next, we have to calculate the projection γ with respect to the decomposition (110). Thus, after simple calculations we obtain that,

$$\gamma\left((x, y, \phi)\overline{x}, \overline{y}, \overline{\phi}, z\right) = \left((x, y, \phi), \lambda_x, \lambda_y, \frac{J}{a}\left(\lambda_y \cos\phi - \lambda_x \sin\phi\right), z\right), \tag{114}$$

where λ_x and λ_y are functions depending on $\left(\phi, \overline{x}, \overline{y}, \overline{\phi}\right)$ given by,

$$\lambda_x = \left(1 - \frac{J}{a^2+J}\sin^2\phi\right)\overline{x} + \frac{J}{a^2+J}\cos\phi\sin\phi\,\overline{y} - \frac{a\sin\phi}{a^2+J}\,\overline{\phi}$$
$$\lambda_y = \left(1 - \frac{J}{a^2+J}\cos^2\phi\right)\overline{y} + \frac{J}{a^2+J}\cos\phi\sin\phi\,\overline{x} - \frac{a\cos\phi}{a^2+J}\,\overline{\phi}$$

Therefore, the contact Eden bracket $\{\cdot,\cdot\}_E$ is given by,

$$\{f, g\}_E = \{f \circ \gamma, g \circ \gamma\},\ \forall f, g \in \mathcal{C}^\infty\left(\mathcal{M} \times \mathbb{R}\right).$$

Recall that the bracket $\{\cdot,\cdot\}$ is the canonical bracket on $T^*Q \times \mathbb{R}$ and, for any two functions $\overline{f}, \overline{g} \in \mathcal{C}^\infty\left(T^*Q \times \mathbb{R}\right)$, we have that,

$$\{\overline{f}, \overline{g}\} = \left(\frac{\partial \overline{g}}{\partial q^i} + p_i\frac{\partial \overline{g}}{\partial z}\right)\frac{\partial \overline{f}}{\partial p_i} - \left(\frac{\partial \overline{f}}{\partial q^i} + p_i\frac{\partial \overline{f}}{\partial z}\right)\frac{\partial \overline{g}}{\partial p_i} + g\frac{\partial \overline{f}}{\partial z} - f\frac{\partial \overline{g}}{\partial z},$$

where, for sake of simplicity, here the canonical coordinates of Q are being denoted by $\left(q^i, p_i\right)$. In this way, we only have to compute the partial derivatives of the composition by γ to obtain the contact Eden bracket for this example.

A remarkable aspect of this example is the simplicity of the resulting contact Eden bracket, especially when compared to the nonholonomic brackets, which have been computed in [24]. Moreover, the computational cost associated with each bracket is significantly different: while the nonholonomic brackets involve a more complex set of calculations (see [24]), the contact Eden bracket presents a substantially lower computational complexity, making it a more efficient alternative in practical applications.

8. Conclusions and further work

Summarizing, we have introduced a general framework of contact nonholonomic dynamics, the *constrained Hamiltonian systems* (Definition 6). For this model, we have established a *geometric constrained Herglotz equation* (34) and its analytical counterpart. (36). Furthermore, we have thoroughly studied several conditions for the existence and uniqueness of solutions. In particular, it is noteworthy that we have found a specific matrix whose rank characterizes the conditions for uniqueness and existence of solutions. Moreover, in such cases, we were able to provide an explicit form of the constrained Hamiltonian vector field by using this matrix along with the Hamiltonian vector field in Eq. (43). As a further step, we provided two additional geometric equations (44) and (45), which are equivalent to the aforementioned Eq. (34).

The next part of the paper is devoted to constructing a *type of bracket for functions* that describe the system's dynamics. First, we construct the nonholonomic brackets through an induced almost Jacobi structure *nonholonomic brackets* by an induced almost Jacobi structure (54). On the other hand, there are still new projections that induce another bracket, referred to as the *P-nonholonomic brackets* (62). We have proved that both brackets are equivalent, and thus, both approaches describe the dynamics of the system.

We have shown that the known case of constraint dynamics for Lagrangian systems studied in [24] can be obtained as a particular case of the current framework. We have also present, by the first time, the Hamiltonian counterpart, which turns out to be another particular case of this framework. Finally, in this context, we have been able to describe another version of the nonholonomic bracket, which is the so-called *contact Eden bracket* (111) and which coincides with the two brackets already described before.

These are some of the subjects on which we wish to work in the near future and which would be a natural continuation of the results contained in this paper:

1. In a future paper we will extend the reduction procedure developed in [10,13] for contact nonholonomic mechanical systems.
2. We want to develop numerical integrators for this type of systems, extending the results of [14,23].
3. We will develop a nonholonomic moment map in this case and the relationship with the nonholonomic bracket [10].

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Data availability

No data was used for the research described in the article.

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